

MESOSCOPIC CENTRAL LIMIT THEOREM FOR NON-HERMITIAN RANDOM MATRICES

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ABSTRACT. We prove that the mesoscopic linear statistics $\sum_i f(n^a(\sigma_i - z_0))$ of the eigenvalues $\{\sigma_i\}_i$ of large $n \times n$ non-Hermitian random matrices with complex centred i.i.d. entries are asymptotically Gaussian for any H_0^2 -functions f around any point z_0 in the bulk of the spectrum on any mesoscopic scale $0 < a < 1/2$. This extends our previous result [13], that was valid on the macroscopic scale, $a = 0$, to cover the entire mesoscopic regime. The main novelty is a *local law* for the product of resolvents for the Hermitization of X at spectral parameters z_1, z_2 with an improved error term in the entire mesoscopic regime $|z_1 - z_2| \gg n^{-1/2}$. The proof is dynamical; it relies on a recursive tandem of the characteristic flow method and the Green function comparison idea combined with a separation of the unstable mode of the underlying stability operator.

1. INTRODUCTION

We consider the eigenvalues $\{\sigma_i\}_{i=1}^n$ of an $n \times n$ random matrix X with i.i.d. entries under the standard normalisation condition $\mathbf{E} x_{ij} = 0$, $\mathbf{E} |x_{ij}|^2 = \frac{1}{n}$. The classical *circular law* [28, 5, 47] asserts that the empirical eigenvalue density converges to the uniform distribution on the unit disk \mathbf{D} :

$$\frac{1}{n} \sum_{i=1}^n f(\sigma_i) \rightarrow \frac{1}{\pi} \int_{\mathbf{D}} f(z) d^2z, \quad n \rightarrow \infty, \quad (1.1)$$

for any continuous bounded test function f . In fact this limit also holds on any mesoscopic scale by the *local circular law* [10], i.e.

$$\frac{n^{2a}}{n} \sum_{i=1}^n f(n^a(\sigma_i - z_0)) \rightarrow \frac{1}{\pi} \int_{\mathbf{C}} f(z) d^2z, \quad 0 < a < \frac{1}{2}, \quad (1.2)$$

where the compactly supported C^2 test function is scaled to concentrate on an n^{-a} -neighborhood of any fixed point in the bulk spectrum $|z_0| < 1$. The threshold $a < 1/2$ is sharp since on scales of order $n^{-1/2}$ there are only finitely many fluctuating eigenvalues hence a law of large number type concentration cannot hold.

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Date: September 8, 2023.

2010 Mathematics Subject Classification. 60B20, 15B52.

Key words and phrases. Dyson Brownian Motion, Local Law, Girko's Formula, Linear Statistics, Central Limit Theorem.

#Supported by ERC Advanced Grant "RMTBeyond" No. 101020331.

*Supported by the SNSF Ambizione Grant PZ00P2 209089.

In this paper we prove a central limit theorem (CLT) for the fluctuation around the local circular law (1.2) for complex i.i.d. matrices, i.e. we show that

$$\sum_{i=1}^n f(n^a(\sigma_i - z_0)) - \frac{n}{n^{2a}} \frac{1}{\pi} \int_{\mathbf{C}} f(z) d^2z - \frac{1}{8\pi} \int_{\mathbf{C}} \Delta f(z) d^2z \implies L(f) \sim \mathcal{N}_{\mathbf{C}}, \quad 0 < a < \frac{1}{2}, \quad (1.3)$$

and compute the variance $\mathbf{E}|L(f)|^2 = \|\nabla f\|^2/4\pi$ of the limiting normal distribution. Note the unusual normalisation: the sum in (1.3) contains roughly n^{1-2a} terms, but it is not divided by $\sqrt{n^{1-2a}}$ unlike for the standard CLT for sums of independent random variables. The eigenvalues σ_i are strongly correlated, and their fluctuations are much smaller than that of an independent point process (e.g. Poisson). It is very remarkable that nevertheless the normal distribution emerges; in fact these eigenvalues asymptotically follow a *Gaussian Free Field (GFF)*, a logarithmically correlated Gaussian process.

The CLT on the macroscopic scale, $a = 0$, around the circular law (1.1),

$$\sum_{i=1}^n f(\sigma_i) - \frac{n}{\pi} \int_{\mathbf{D}} f(z) d^2z + \frac{\kappa_4}{\pi} \int_{\mathbf{D}} f(z)(2|z|^2 - 1) d^2z - \frac{1}{8\pi} \int_{\mathbf{C}} \Delta f(z) d^2z \implies L(f) \sim \mathcal{N}_{\mathbf{C}}, \quad (1.4)$$

has been proven earlier with a long history. Here $\kappa_4 := n^2[\mathbf{E}|x_{ij}|^4 - 2(\mathbf{E}|x_{ij}|^2)^2 - |\mathbf{E}x_{ij}^2|^2]$ denotes the normalised joint cumulant of $x_{ij}, x_{ij}, \bar{x}_{ij}, \bar{x}_{ij}$. Historically the first results were for the *complex Ginibre ensemble*, i.e. when x_{ij} are complex Gaussians (and therefore $\kappa_4 = 0$); in this case an explicit formula for the joint density function of all eigenvalues is available. Forrester in [26] proved (1.4) for radially symmetric f , he found $\mathbf{E}|L(f)|^2 = (4\pi)^{-1} \int_{\mathbf{D}} |\nabla f|^2 d^2z$ and gave a heuristic prediction that the variance $\mathbf{E}|L(f)|^2$ contains an additional boundary term for general f . Rider and Virág in [43] have rigorously verified Forrester's prediction for any $f \in C^1(\mathbf{D})$ and they also presented a GFF interpretation of the result. Rider in [41] also considered special indicator test functions depending only on the angle or on the modulus that are not in $H^1(\mathbf{D})$ even in the mesoscopic regime with $\mathbf{E}|L(f)|^2$ growing as $\log n$.

Beyond the explicitly computable complex Ginibre case the first result was obtained by Rider and Silverstein [42, Theorem 1.1]. They proved (1.4) for X with general i.i.d. complex matrix elements but only for test functions f that are analytic on an unnaturally large disk of radius 2. In the real symmetry class the domain of analyticity was optimized in [40] and the result was also extended to *elliptic ensembles* allowing correlation between x_{ij} and x_{ji} . Later even *products* of i.i.d. matrices were considered in [22]. Alternatively, the moment method was used in [39] to prove CLT with polynomial test functions f . Beyond analytic test functions, Nguyen and Vu in [38] proved a CLT for $f = \log$ and X with i.i.d. entries; Tao and Vu in [48, Corollary 10] proved CLT for the counting function on balls even on mesoscopic scales assuming the first four moments of X match those of the complex Ginibre ensemble. The comparison method from Tao and Vu was extended by Kopel [32, Corollary 1] to general smooth test functions and also to real X with an additional study on the real eigenvalues (see also [46]). Finally, the macroscopic CLT (1.4) in full generality, i.e. for matrices with general i.i.d. entry distribution and general smooth (in fact $H^{2+\delta}$) test functions, has been proven by us in [13] and in [16] for the complex and real cases, respectively. For more details on the history of the circular laws (1.1)-(1.2) and the macroscopic CLT (1.4), as well as several further references, see [13, 16].

Apart from [41] that holds only for complex Ginibre with special test functions and apart from [48, Corollary 10] that assumes four matching moments and test functions being the indicator functions of mesoscopic balls, all previous CLT results were on macroscopic scales. The extension to mesoscopic scales (1.3) is clearly not feasible with moment method or with methods relying on the analyticity of the test function. Direct calculations based upon the explicit Ginibre formulas beyond the special test functions in [41, 48] may be possible also for mesoscopic scales but their extension to the general i.i.d. case would again require four moment matching.

In this paper we demonstrate that our approach in [13, 16] can be extended to cover the entire mesoscopic regime. For simplicity, we work with the complex symmetry class, the real case requires additional technical steps that we will explain in Remark 3.6 but do not carry out in details. To highlight the novelty of the current proof, we briefly recall the main ideas in [13]. The starting point is Girko's formula,

$$\sum_{\sigma \in \text{Spec}(X)} f(\sigma) = -\frac{1}{4\pi} \int_{\mathbf{C}} \Delta f(z) \int_0^\infty \Im \text{Tr} G^z(i\eta) d\eta d^2z, \quad (1.5)$$

expressing the linear statistics of the non-Hermitian eigenvalues of X in terms of the resolvent $G^z(w) := (H^z - w)^{-1}$ of the *Hermitisation* H^z of X ,

$$H^z := \begin{pmatrix} 0 & X - z \\ X^* - \bar{z} & 0 \end{pmatrix} \quad (1.6)$$

parametrized by $z \in \mathbf{C}$. Note that Girko's formula requires to understand the resolvent $G^z(i\eta)$ very well for arbitrary small η , even for the macroscopic CLT.

The different η -regimes in (1.5) require very different methods.

Sub-critical $\eta \ll 1/n$: The absence of eigenvalues of H^z in $[-\eta, \eta]$ is proved using smoothing inequalities for the lower tail of the lowest eigenvalue [44, 49, 50] (see also [12, 18, 45]). This regime is handled exactly as in the macroscopic case [13, 16].

Critical $\eta \sim 1/n$: The asymptotic independence of resolvents for $|z_1 - z_2| \gg 1/\sqrt{n}$ is proved dynamically using the Dyson Brownian motion (DBM) technique. Our new result is the asymptotic orthogonality of the low-lying eigenvectors of H^{z_1} and H^{z_2} . Once this key input is established, the proof proceeds exactly as in [16] following the coupling technique and the homogenization idea that were first introduced in [8] for Wigner matrices, substantially generalized later in [35] for general DBM (see also [7]), and adapted to H^z in [11]. The almost orthogonality of eigenvectors implies the almost independence of the driving Brownian motions in the DBM for λ^{z_1} and λ^{z_2} .

Super-critical $\eta \gg 1/n$: A central limit theorem for resolvents is established using iterated cumulant expansions. The covariance of $\text{Tr } G^{z_1}(i\eta_1)$ and $\text{Tr } G^{z_2}(i\eta_2)$ for *different* parameters $z_1 \neq z_2$ depends critically on the product of resolvents $\text{Tr } G^{z_1}(i\eta_1)G^{z_2}(i\eta_2)$ which we evaluate to high precision using our new *multi-resolvent local laws*.

We now explain the novelty of the present work for the critical and super-critical regime in form of an improved multi-resolvent local law. The conventional single resolvent local law asserts that $G^z(i\eta)$ can be approximated deterministically by an explicitly computable matrix $M^z(i\eta)$ (see (3.4)) up to a negligible error as long as $|\eta| \gg 1/n$:

$$\frac{1}{2n} \left| \text{Tr} [G^z(i\eta) - M^z(i\eta)] \right| \lesssim \frac{n^\xi}{n|\eta|} \quad (1.7)$$

holds with very high probability for any fixed $\xi > 0$.

While single resolvent local laws are well understood, their multi-resolvent versions are much more subtle. The naive intuition from (1.7) would suggest that $G^{z_1}(i\eta_1)G^{z_2}(i\eta_2) \approx M^{z_1}(i\eta_1)M^{z_2}(i\eta_2)$, but this is wrong. The correct deterministic approximation of $G^{z_1}(i\eta_1)G^{z_2}(i\eta_2)$ is $M_{12} := \mathcal{B}_{12}^{-1}[M^{z_1}(i\eta_1)M^{z_2}(i\eta_2)]$, where \mathcal{B}_{12} is the *stability operator*, given explicitly in (3.13). This operator has a small eigenvalue β of order $|z_1 - z_2|^2 + \eta_1 + \eta_2$. The key question is the error term in this approximation.

Setting $\eta_1 = \eta_2 =: \eta > 0$ for simplicity, one may guess (and we prove below) the bound

$$\frac{1}{2n} \left| \text{Tr} [G^{z_1}(i\eta)G^{z_2}(i\eta) - M_{12}] \right| \lesssim \frac{n^\xi}{n\eta^2}. \quad (1.8)$$

If $z_1 = z_2$, then this bound is essentially optimal and in this case $M_{12} \sim 1/\eta$ since $\beta \sim \eta$. However, when f in (1.5) is mesoscopically supported, then typically we have $|z_1 - z_2| \sim n^{-a} \gg n^{-1/2}$ in the calculation of the variance of (1.5). In this case $M_{12} \sim [|z_1 - z_2|^2 + \eta]^{-1}$, i.e. for η 's such that $1/n \ll \eta \ll |z_1 - z_2|^2 \sim n^{-2a}$, the bound on M_{12} is already smaller than $1/\eta$. One therefore expects that the error term (1.8) also improves in this regime and indeed we need some improvement to handle the $\eta \gg 1/n$ regime when computing higher moments of (1.5). The main technical result in our proofs of the macroscopic CLT was [13, Theorem 5.2], asserting that the error in (1.8) improves by a factor $n^{-\epsilon_1}$ if $|z_1 - z_2| \geq n^{-\epsilon_2}$ for some small ϵ_1, ϵ_2 . This improvement, however, does not apply to genuine mesoscopic scales when $|z_1 - z_2| \sim n^{-a} \ll n^{-\epsilon_2}$. In Theorem 3.3 of this paper we present a substantial improvement of [13, Theorem 5.2], essentially asserting that the error in (1.8) can be improved by a factor $n^{-\epsilon_1}$ as long as $|z_1 - z_2| \geq n^{-\frac{1}{2} + \epsilon_2}$, i.e. the improvement is present in the entire mesoscopic regime.

We stress that (1.8) even without any additional improvement is new in the mesoscopic regime and its proof is highly nontrivial. Multi-resolvent local laws of the form (1.8) for products of resolvents $G(z) = (W - z)^{-1}$ of Wigner matrices W at different spectral parameters have been proven earlier (see e.g. [21, Theorem 3.4] and its refinements in [15, Proposition 3.4], [17, Proposition 5.1], [19, Theorem 2.5], and [20, Theorem 2.2]). However, these proofs rely on the fact that both resolvents stem from the *same* Hermitian matrix hence they have the same spectral resolution. This allowed us to use resolvent identities to express products of resolvents in terms of their first powers and thus prevent the instability of the operator \mathcal{B}_{12} from influencing the error terms. Practically, multi-resolvent local laws were reduced to well established

single resolvent local laws by algebraic identities[†]. The spectral resolution of H^{z_1} and H^{z_2} , however, are different even though they are defined via the same matrix X . Lacking the convenient resolvent identity, the previous methods would involve inverting \mathcal{B}_{12} even along its unstable direction which would lead to an error term that is bigger than (1.8) by a large multiplicative factor $|\beta|^{-1}$. This was still affordable in the proof of the macroscopic CLT since in the typical regime we had $|z_1 - z_2| \sim 1$, hence $|\beta|^{-1} \sim 1$ was harmless. The mesoscopic regime requires a completely new approach, which allows us to deal with the unstable direction of the stability operator in a novel way completely circumventing the resolvent identity.

The proof of our improved multi-resolvent local law relies on the *characteristic flow method* that has previously been used for single-resolvent local laws and closely related quantities in various models in [30, 1, 33, 2, 34, 9, 7]. To our best knowledge, this method has not been applied in a multi-resolvent setup before with the exception of [9, Proposition 4.5] where the product of special time-evolved resolvents at two different times was considered.

The key idea in the characteristic flow method is to consider an Ornstein-Uhlenbeck flow X_t with initial condition $X_0 = X$ and follow the time evolution of its Hermitised resolvent G_t , but at time evolved spectral parameters z_t and η_t . These parameters satisfy a natural first order differential equation (the characteristic flow equation) that is chosen so that the leading terms in the flow $\text{Tr} G_t^{z_{1,t}}(i\eta_{1,t})G_t^{z_{2,t}}(i\eta_{2,t})$ cancel out. In particular, unlike previous results using the characteristic flow method, we consider a *matrix version of the characteristics* (see (A.18)) for the first time in the random matrix setting, which we believe to be useful also for much more general random matrix ensembles. Along this flow $\eta_t > 0$ decreases. In this way one can transfer a local law (1.8) from large $\eta > 0$ to a similar local law for much smaller $\eta > 0$ at the expense of adding a Gaussian component to the entry distribution of X but without an additional $|\beta|^{-1}$ factor, see Proposition 5.3. Furthermore, when η becomes smaller than the threshold $|z_1 - z_2|^2$, one $\eta^{-1/2}$ factor in the right hand side of (1.8) switches to the better $|\beta|^{-1/2} \sim |z_1 - z_2|^{-1}$ factor, explicitly bringing in the improvement we were looking for. Next, we need to remove the added Gaussian component by fairly standard Green function comparison arguments, however maintaining the improved precision in the error term requires nontrivial extra work. Technically we do this via a *recursive tandem*: we successively reduce η by small steps and we immediately remove the Gaussian component before further reduction in η . The regimes $\eta \gtrsim |z_1 - z_2|^2$ and $\eta \lesssim |z_1 - z_2|^2$ need separate estimates.

Remark 1.1. *We mention that with a similar (in fact simpler) scheme it is possible to give an alternative short proof of the single resolvent local law for $G^z(i\eta)$ as well, see (3.6) later. The recursive tandem of the characteristic flow and the GFT offers an alternative proof to many existing optimal local laws and it seems to be more powerful than previous methods in many situations. For the sake of brevity of the current paper we refrain from demonstrating this idea for $G^z(i\eta)$ since the optimal local law in this case has already been established.*

The multi-resolvent local law Theorem 3.3, the improved version of the local law (1.8), is also the key novel input for the critical η -regime and implies the almost orthogonality of eigenvectors w_i^z of H^z in the entire mesoscopic regime:

$$|\langle w_i^{z_1}, w_j^{z_2} \rangle| \leq n^{-\epsilon_1}, \quad |z_1 - z_2| \gg n^{-1/2 + \epsilon_2}, \quad (1.9)$$

for any i, j , and a similar bound for the singular vectors of $X - z$, see Lemma 3.1. The analogous result was proven only for $|z_1 - z_2| \gg n^{-\epsilon}$ in [13, 16].

We remark that the relation (1.9) together with the closely related almost independence of the low lying eigenvalues λ^{z_1} and λ^{z_2} are of substantial interest in themselves, independently of the proof of the mesoscopic CLT. Since low lying eigenvalues of H^z are intuitively related to the eigenvalues of X near z , the above relations indicate that the local spectral behavior of X near z_1 and z_2 are largely independent as long as $|z_1 - z_2| \gg n^{-1/2}$, in other words the ‘‘correlation length’’ in the spectrum of X is $n^{-1/2}$. We stress that currently we do not know how to turn this intuition into a rigorous proof for general i.i.d. ensemble since our results concern only the eigenvalues and eigenvectors of the Hermitisation of H^z , equivalently, the singular values and singular vectors of $X - z$ for any fixed (deterministic) z , and not directly the eigenvalues and eigenvectors of X .

We note that for Ginibre ensemble the explicit two point eigenvalue correlation function exhibits an exponential decay for $|z_1 - z_2| \gg n^{-1/2}$ and the *eigenvector overlap* of eigenvectors belonging to two different eigenvalues of X has been explicitly computed in [27, 6], effectively proving a polynomial decay of correlation of eigenvectors beyond the scale $n^{-1/2}$ in the spectrum of X . The extension of these results to i.i.d. matrices remains an outstanding open problem,

[†]The analysis was more complicated when deterministic matrices A_1, A_2, \dots were also present, i.e. we considered local laws for $G_1 A_1 G_2 A_2 \dots$. In this case we split A 's into a diagonal and a traceless part; the former was handled by resolvent identities, while on traceless matrices the stability operator has bounded inverse. In nutshell, we still circumvented the instability of \mathcal{B}_{12}^{-1} by resolvent identities in the Wigner case.

closely related to the unsolved *bulk universality conjecture*² for the local eigenvalue statistics of an i.i.d. matrix X , which represents the non-Hermitian analogue of the celebrated Wigner-Dyson-Mehta universality for Wigner matrices [25].

Notations and conventions. We introduce some notations we use throughout the paper. For integers $k \in \mathbf{N}$ we use the notation $[k] := \{1, \dots, k\}$. We write $\mathbf{D} \subset \mathbf{C}$ for the open unit disk, and for any $z \in \mathbf{C}$ we use the notation $d^2z := 2^{-1}i(dz \wedge d\bar{z})$ for the two dimensional volume form on \mathbf{C} . For positive quantities f, g we write $f \lesssim g$ and $f \sim g$ if $f \leq Cg$ or $cg \leq f \leq Cg$, respectively, for some constants $c, C > 0$ which depend only on the constants appearing in (2.1) and on a, τ in (2.3). For any two positive real numbers $\omega_*, \omega^* \in \mathbf{R}_+$ by $\omega_* \ll \omega^*$ we denote that $\omega_* \leq c\omega^*$ for some small constant $0 < c \leq 1/100$. We denote vectors by bold-faced lower case Roman letters $\mathbf{x}, \mathbf{y} \in \mathbf{C}^k$, for some $k \in \mathbf{N}$. Vector and matrix norms, $\|\mathbf{x}\|$ and $\|A\|$, indicate the usual Euclidean norm and the corresponding induced matrix norm. For any $d \times d$ matrix A we use the notation $\langle A \rangle := \frac{1}{d} \text{Tr } A$ to denote the normalized trace of A . Moreover, for vectors $\mathbf{x}, \mathbf{y} \in \mathbf{C}^n$ and matrices $A, B \in \mathbf{C}^{2n \times 2n}$ we define the scalar product

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_i \bar{x}_i y_i, \quad \langle A, B \rangle := \langle A^* B \rangle.$$

For an open set $\Omega \subset \mathbf{C}$, by $H_0^2(\Omega)$ we denote the Sobolev space defined as the completion of the smooth compactly supported functions $C_c^\infty(\Omega)$ under the norm

$$\|f\|_{H_0^2(\Omega)} = \left(\int_\Omega |\nabla f(z)|^2 d^2z \right)^{1/2}.$$

We will use the concept of “with very high probability” meaning that for any fixed $D > 0$ the probability of the event is bigger than $1 - n^{-D}$ if $n \geq n_0(D)$. Moreover, we use the convention that $\xi > 0$ denotes an arbitrary small exponent which is independent of n . We introduce the notion of *stochastic domination* (see e.g. [23]): given two families of non-negative random variables

$$X = \left(X^{(n)}(u) \mid n \in \mathbf{N}, u \in U^{(n)} \right) \quad \text{and} \quad Y = \left(Y^{(n)}(u) \mid n \in \mathbf{N}, u \in U^{(n)} \right)$$

indexed by n (and possibly some parameter u in some parameter space $U^{(n)}$), we say that X is stochastically dominated by Y , if for all $\xi, D > 0$ we have

$$\sup_{u \in U^{(n)}} \mathbf{P} \left[X^{(n)}(u) > n^\xi Y^{(n)}(u) \right] \leq n^{-D} \quad (1.10)$$

for large enough $n \geq n_0(\xi, D)$. In this case we use the notation $X \prec Y$ or $X = \mathcal{O}_\prec(Y)$.

Acknowledgement. The authors are grateful to Joscha Henheik for his help with the formulas in Appendix B.

2. MAIN RESULT

Let X be an $n \times n$ matrix with independent identically distributed (i.i.d.) complex entries such that $x_{ab} \stackrel{d}{=} n^{-1/2} \chi$ for some complex random variable χ , satisfying the following:

Assumption 1. *The random variable χ satisfies $\mathbf{E} \chi = \mathbf{E} \chi^2 = 0$ and $\mathbf{E} |\chi|^2 = 1$. In addition we assume the existence of its high moments, i.e. that there exist constants $C_p > 0$, for any $p \in \mathbf{N}$, such that*

$$\mathbf{E} |\chi|^p \leq C_p. \quad (2.1)$$

We focus on the complex symmetry class, see Remark 3.6 for the necessary modifications for the real case.

Denote by $\{\sigma_i\}_{i \in [n]}$ the eigenvalues of X , and consider the centered linear statistics

$$L_n(f_{z_0, a}) := \sum_i f_{z_0, a}(\sigma_i) - \mathbf{E} \sum_i f_{z_0, a}(\sigma_i), \quad (2.2)$$

with

$$f_{z_0, a}(z) := f(n^a(z - z_0)), \quad a \in \left(0, \frac{1}{2}\right), \quad |z_0| \leq 1 - \tau, \quad (2.3)$$

for some small fixed $\tau > 0$. Here $f \in H_0^2(\Omega)$ with a compact set $\Omega \subset \mathbf{C}$. We already mentioned in the introduction that the sum in $L_n(f_{z_0, a})$ contains roughly n^{1-2a} summands but it is not normalized by $(n^{1-2a})^{-1/2}$ unlike in the standard CLT for independent summands.

²The same universality at the edge of the spectrum along the unit circle has been proven in [14].

The main result of this paper is the following Central Limit Theorem for all mesoscopic scales.

Theorem 2.1. *Let X be an $n \times n$ matrix satisfying Assumption 1, fix a small $\tau > 0$, and let $|z_0| \leq 1 - \tau$ and $a \in (0, \frac{1}{2})$. Let $f_{z_0, a}$ be defined as in (2.3), with $f \in H_0^2(\Omega)$ for some a compact set $\Omega \subset \mathbf{C}$. Then $L_n(f_{z_0, a})$ converges (in the sense of moments³ and therefore in distribution) to a complex Gaussian random variable $L(f)$,*

$$L_n(f_{z_0, a}) \xrightarrow{\text{m}} L(f),$$

with expectation $\mathbf{E} L(f) = 0$, and second moments $\mathbf{E} |L(f)|^2 = C(f, f)$, $\mathbf{E} L(f)^2 = C(\bar{f}, f)$, where

$$C(g, f) := \frac{1}{4\pi} \langle \nabla g, \nabla f \rangle_{L^2(\Omega)}. \quad (2.4)$$

Moreover, the expectation is given by

$$\mathbf{E} \sum_i f_{z_0, a}(\sigma_i) = \frac{n^{1-2a}}{\pi} \int_{\mathbf{C}} f(z) \, d^2z + \frac{1}{8\pi} \int_{\mathbf{C}} \Delta f(z) \, d^2z + \mathcal{O}(n^{-2a} + n^{-c}), \quad (2.5)$$

for some small fixed $c > 0$. The implicit constant in $\mathcal{O}(\cdot)$ may depend on $\|\Delta f\|_{L^2(\Omega)}$, and $|\Omega|$.

Remark 2.2.

- (i) In Theorem 2.1 we stated the CLT for the mesoscopic regime $a \in (0, 1/2)$. The complementary regime $a \in [0, \epsilon]$, for some small fixed $\epsilon > 0$, was already covered in [13, Remark 2.6] (in fact that proof also holds on any almost macroscopic scale where the power function n^a in (2.3) is replaced with any sequence $C_n \rightarrow \infty$, $C_n \leq n^\epsilon$).
- (ii) Note that in the macroscopic regime, $a = 0$, both asymptotic variance and expectation of $L_n(f)$ depend on the fourth cumulant $\kappa_4 = \kappa_4(\chi) = \mathbf{E}|\chi|^4 - 2(\mathbf{E}|\chi|^2)^2$ of the entry distribution, c.f. [13, Theorem 2.2]. The fact that in the mesoscopic regime the CLT only depends on the first two moments of χ is yet another manifestation of the effect that local properties of the spectrum are more universal than global properties.

Remark 2.3 (Gaussian Free Field). We recall from [13, Section 2.1] that in the macroscopic case $a = 0$ for $\kappa_4 \geq 0$ the limiting Gaussian process L could be interpreted as

$$L = \frac{1}{\sqrt{4\pi}} Ph + \sqrt{\kappa_4} (\langle \cdot \rangle_{\mathbf{D}} - \langle \cdot \rangle_{\partial \mathbf{D}}) \Xi, \quad (2.6)$$

where Ph is the projection of the Gaussian free field (GFF) h on some bounded domain $\Omega \supset \mathbf{D}$ conditioned to be harmonic in the complement \mathbf{D}^c of the unit disk \mathbf{D} , Ξ is a standard real Gaussian variable, independent of h , and $\langle \cdot \rangle_{\mathbf{D}}$, $\langle \cdot \rangle_{\partial \mathbf{D}}$ denote the averaging functionals on \mathbf{D} , $\partial \mathbf{D}$.

By Theorem 2.1 we can now conclude that in the bulk on all mesoscopic scales $0 < a < 1/2$,

$$L = \frac{1}{\sqrt{4\pi}} h \quad (2.7)$$

for some h in the equivalence class of the whole-plane GFF (which is only defined modulo additive constants), see e.g. [37, Section 2.2.1]. Compared to the macroscopic case, however, this interpretation is also valid for $\kappa_4 < 0$.

Remark 2.4. By polarization we also conclude a multivariate CLT. In fact, our proof actually gives the joint Gaussianity of

$$L_n(f_{z_1, a_1}^{(1)}), L_n(f_{z_2, a_2}^{(2)}), \dots, L_n(f_{z_p, a_p}^{(p)}), \quad p \in \mathbf{N},$$

for the rescaled versions $f_{z_i, a_i}^{(i)}(z) := f^{(i)}(n^{a_i}(z - z_i))$ of different test functions $f^{(i)} \in H_0^2(\Omega)$ around fixed reference points z_i with $|z_i| \leq 1 - \tau$ and scales $a_i \in (0, 1/2)$ that are not necessarily equal. In particular, from our proof below (especially around (4.4)) one can easily see that $\mathbf{E} L_n(f_{z_i, a_i}^{(i)}) \overline{L_n(f_{z_j, a_j}^{(j)})}$ can be non-zero (in the limit $n \rightarrow \infty$) only if $a_i = a_j$, $z_i = z_j$ and $\text{supp}(f^{(i)}) \cap \text{supp}(f^{(j)}) \neq \emptyset$.

Remark 2.5. In Theorem 2.1 we assumed $f \in H_0^2(\Omega)$ for simplicity, but in fact the somewhat weaker regularity condition of the form $\text{supp}(f) \subset \Omega$ with $\Delta f \in L^{1+\delta}(\Omega)$ for any fixed $\delta > 0$ and compact Ω suffices. By the Calderon-Zygmund inequality, this condition implies $f \in W_0^{2, 1+\delta}(\Omega)$, hence $f \in H^1(\Omega)$ as well, and these conditions are sufficient for our proof. Note that the assumption $f \in H^1(\Omega)$ is necessary to make sure that the variance in (2.4) is finite; in particular, this implies that our regularity condition on f is close to being optimal.

³We say that a sequence of random variables Y_n converges to Y_∞ in the sense of moments, $Y_n \xrightarrow{\text{m}} Y_\infty$, if $\mathbf{E} |Y_n|^k = \mathbf{E} |Y_\infty|^k + \mathcal{O}(n^{-c(k)})$ for any $k \in \mathbf{N}$ for some small $c(k) > 0$. Even if not stated explicitly, the implicit constant in $\mathcal{O}(\cdot)$ may depend on k .

3. PROOF STRATEGY

To analyze the linear statistics in (2.2) for the test function $f_{z_0,a}$ given in (2.3), we rely on Girko's formula [28] in the form used in [48]:

$$\begin{aligned} L_n(f_{z_0,a}) &= \frac{1}{4\pi} \int \Delta f_{z_0,a}(z) \left[\log |\det(H^z - iT)| + i \left(\int_0^{\eta_0} + \int_{\eta_0}^{\eta_c} + \int_{\eta_c}^T \right) \left[\text{Tr}[G^z(i\eta)] - \mathbf{E} \text{Tr}[G^z(i\eta)] \right] d\eta \right] d^2z \\ &=: J_T(f_{z_0,a}) + I_0^{\eta_0}(f_{z_0,a}) + I_{\eta_0}^{\eta_c}(f_{z_0,a}) + I_{\eta_c}^T(f_{z_0,a}), \end{aligned} \quad (3.1)$$

where we choose the integration thresholds

$$\eta_0 := n^{-1-\delta_0}, \quad \eta_c := n^{-1+\delta_1},$$

for some small $\delta_0, \delta_1 > 0$ and we set $T = n^{100}$. We recall the Hermitisation H^z of $X - z$ given by

$$H^z := \begin{pmatrix} 0 & X - z \\ (X - z)^* & 0 \end{pmatrix}, \quad (3.2)$$

and its resolvent $G^z(w) := (H^z - w)^{-1}$, with $w \in \mathbf{C} \setminus \mathbf{R}$. By spectral symmetry, the eigenvalues of H^z come in opposite pairs, so we label them as $\{\lambda_{\pm i}^z\}_{i \in [n]}$ where $\lambda_{-i}^z = -\lambda_i^z$. The corresponding orthonormal eigenvectors $\{\mathbf{w}_{\pm i}^z\}_{i \in [n]}$ consequently decompose into two n -vectors with symmetry $\{\mathbf{w}_{\pm i}^z\}_{i \in [n]} = \{(\mathbf{u}_i^z, \pm \mathbf{v}_i^z)\}_{i \in [n]}$ and with $\|\mathbf{u}_i^z\|^2 = \|\mathbf{v}_i^z\|^2 = 1/2$.

We have split (3.1) into the sum of four terms and each of them will be analyzed using different techniques. The term J_T will easily be estimated as in [4, Proof of Theorem 2.3], whilst the fact that $I_0^{\eta_0}$ is negligible will follow by smoothing inequalities for the smallest singular value of $X - z$ (see [49, Theorem 3.2]). To estimate $I_{\eta_0}^{\eta_c}$, which is the regime when η is proportional to the level spacing of the eigenvalues of H^z around zero, we will need the asymptotic independence of $\text{Tr}G^{z_1}$ and $\text{Tr}G^{z_2}$ for $|z_1 - z_2| \gg n^{-1/2}$ (see Proposition 3.5 below). The proof of this proposition is analogous to [13, Section 7] relying on the Dyson Brownian motion, once the following asymptotic orthogonality of the singular vectors of $X - z_1$ and $X - z_2$ is proven.

Theorem 3.1. *Let $\{\mathbf{w}_{\pm i}^{z_l}\}_{i=1}^n = \{(\mathbf{u}_i^{z_l}, \pm \mathbf{v}_i^{z_l})\}_{i=1}^n$ be the eigenvectors of H^{z_l} for $l = 1, 2$. Then for any sufficiently small $\omega_d, \omega_p > 0$ there exist $\omega_B, \omega_E > 0$ such that if $n^{-1/2+\omega_p} \leq |z_1 - z_2| \leq n^{-\omega_d}$, then*

$$|\langle \mathbf{u}_i^{z_1}, \mathbf{u}_j^{z_2} \rangle| + |\langle \mathbf{v}_i^{z_1}, \mathbf{v}_j^{z_2} \rangle| \leq n^{-\omega_E}, \quad 1 \leq i, j \leq n^{\omega_B}, \quad (3.3)$$

with very high probability.

The proof of this theorem is presented in Section A.2; it will be a simple consequence of the new two-resolvent local law for $\text{Tr}G^{z_1}(i\eta_1)G^{z_2}(i\eta_2)$ given in Theorem 3.3 below. Finally, the leading contribution to $L_n(f)$ comes from the regime $I_{\eta_c}^T$, which thus needs to be computed more precisely by identifying all its moments. This is done first for the resolvent in Proposition 3.4 and later we extend it to the z - and η -integrals of the resolvent according to Girko's formula. The fundamental input to prove Proposition 3.4 will be again our new local law from Theorem 3.3.

Before stating the new two-resolvent local law for $\text{Tr}G^{z_1}G^{z_2}$ we recall the local law for a single resolvent. In the regime $|\Im w| \gg 1/n$ the resolvent $G^z(w)$ has a deterministic leading term. This deterministic approximation is given by

$$M^z(w) := \begin{pmatrix} m^z(w) & -zu^z(w) \\ -\bar{z}u^z(w) & m^z(w) \end{pmatrix}, \quad u^z(w) := \frac{w}{w + m^z(w)}, \quad (3.4)$$

with m^z being the unique solution of the scalar equation, with a side condition,

$$-\frac{1}{m^z(w)} = w + m^z(w) - \frac{|z|^2}{w + m^z(w)}, \quad \Im m^z(w) \Im w > 0. \quad (3.5)$$

Here $M^z = M^z(w)$ is a $2n \times 2n$ block constant matrix, i.e. it has a 2×2 block structure with each block being a constant multiple of the $n \times n$ identity. We remark that throughout this paper by 2×2 block matrices we always refer to $2n \times 2n$ matrices which consists of four $n \times n$ blocks. Furthermore, we will say that a matrix $A \in \mathbf{C}^{2n \times 2n}$ is block traceless if

it is a 2×2 block matrix such that the trace of each of its blocks is equal to zero. We will work on the imaginary axis, $\Re w = 0$, as required in Girko's formula. More precisely, we have the following *average* and *isotropic* local laws from [3]

$$|\langle A(G^z(i\eta) - M^z(i\eta)) \rangle| \prec \frac{\|A\|}{n|\eta|}, \quad |\langle \mathbf{x}, (G^z(i\eta) - M^z(i\eta))\mathbf{y} \rangle| \prec \frac{\|\mathbf{x}\| \|\mathbf{y}\|}{\sqrt{n|\eta|}} \quad (3.6)$$

for any deterministic matrix and vectors $A, \mathbf{x}, \mathbf{y}$, uniformly in $|\eta| \geq n^{-1+\epsilon}$ and $|z| \leq 1 - \tau$. Furthermore, using trivial computations, along the imaginary axis we have the expansion for m^z, u^z :

Lemma 3.2. *Fix small $\eta, \tau > 0$, z such that $|z| \leq 1 - \tau$, let $m^z(i\eta)$ be the unique solution of (3.5), and let $u^z(i\eta)$ be defined as in (3.4). Then, we have*

$$\begin{aligned} m^z(i\eta) &= i\sqrt{1 - |z|^2} + i\eta \frac{2|z|^2 - 1}{2(1 - |z|^2)} + \mathcal{O}(\eta^2), \\ u^z(i\eta) &= 1 - \frac{\eta}{\sqrt{1 - |z|^2}} + \mathcal{O}(\eta^2). \end{aligned} \quad (3.7)$$

The implicit constant in $\mathcal{O}(\cdot)$ depends on τ .

The deterministic approximation M^z comes from the unique solution of the *matrix Dyson equation*

$$-[M^z(w)]^{-1} = \begin{pmatrix} w & z \\ \bar{z} & w \end{pmatrix} + \mathcal{S}[M^z(w)], \quad w \in \mathbf{C} \setminus \mathbf{R}$$

with the side condition $(\Im w)\Im M^z(w) > 0$. Here \mathcal{S} is the *covariance operator* which acts on any matrix $R \in \mathbf{C}^{2n \times 2n}$ as⁴

$$\mathcal{S}[R] := \begin{pmatrix} \langle R_{22} \rangle & 0 \\ 0 & \langle R_{11} \rangle \end{pmatrix} \quad \text{i.e.} \quad \mathcal{S}[R] = 2\langle RE_2 \rangle E_1 + 2\langle RE_1 \rangle E_2 = \langle R \rangle - \langle RE_- \rangle E_-, \quad (3.8)$$

where R_{ij} , with $i, j \in [2]$ are the four blocks of R . In the second formula we expressed \mathcal{S} in a basis representation, where we set $E_- := E_1 - E_2$ and we defined the 2×2 block constant matrices

$$E_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad F := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (3.9)$$

Note that $\{\sqrt{2}E_1, \sqrt{2}E_2, \sqrt{2}F, \sqrt{2}F^*\}$ is an orthonormal basis of the 2×2 block constant matrices.

The *averaged (tracial) version* of our key two-resolvent local contained in the following theorem.

Theorem 3.3. *Fix small $\epsilon, \tau, \omega_d > 0$, independent of n and let $z_1, z_2 \in \mathbf{C}$ such that $|z_i| \leq 1 - \tau$ and $|z_1 - z_2| \leq n^{-\omega_d}$, then for any deterministic matrices A, B it holds*

$$|\langle (G^{z_1}(i\eta_1)AG^{z_2}(i\eta_2) - M_{12}^A)B \rangle| \prec \frac{1}{n\eta_*^{3/2}(\eta^*)^{1/2}} \left(\eta_*^{1/6} + n^{-1/10} + \frac{1}{\sqrt{n\eta_*}} + \left(\frac{\eta^*}{\eta^* + |z_1 - z_2|^2} \right)^{1/4} \right), \quad (3.10)$$

with

$$M_{12}^A = M_{12}^A(z_1, i\eta_1, z_2, i\eta_2) := (1 - M_1 \mathcal{S}[\cdot] M_2)^{-1} [M_1 A M_2], \quad M_j := M^{z_j}(i\eta_j), \quad j = 1, 2, \quad (3.11)$$

and $\eta_* := |\eta_1| \wedge |\eta_2|$, $\eta^* := |\eta_1| \vee |\eta_2|$. The bound in (3.10) holds uniformly for any matrices A, B with $\|A\| + \|B\| \lesssim 1$ and for $\eta_* \geq n^{-1+\epsilon}$. Additionally, for the deterministic term in (3.10) we have the bound

$$\|M_{12}^A\| \lesssim \frac{1}{|z_1 - z_2|^2 + \eta^*}. \quad (3.12)$$

The proof of Theorem 3.3 will be divided into two steps: (i) in Section 5 we use the characteristics flow (see the introduction for relevant references using this method) to show that if we know (3.10) for large η 's then we can propagate the same bound to smaller η 's at the expense of adding a Gaussian component to X (this will be used with an initial $\eta \sim 1$), (ii) in Section 6 we use a *Green function comparison argument (GFT)* to remove the Gaussian component added in (i).

⁴Here we recall the convention that for any $A \in \mathbf{C}^{d \times d}$ we use the notation $\langle A \rangle = d^{-1} \text{Tr}[A]$. For example, $\langle R \rangle = (2n)^{-1} \text{Tr}[R]$ and $\langle R_{11} \rangle = n^{-1} \text{Tr}[R_{11}]$.

As we already pointed out in the introduction, the local law in (3.10) is a significant improvement compared to [13, Theorem 5.2]. The main difference is that the error term in [13, Theorem 5.2] contained an additional large factor $\|\mathcal{B}_{12}^{-1}\|$, the norm of the inverse of the *stability operator*

$$\mathcal{B}_{12} := 1 - M_1 S[\cdot] M_2 \quad (3.13)$$

acting on $\mathbf{C}^{2n \times 2n}$ matrices. As we have $\|\mathcal{B}_{12}^{-1}\| \sim [|z_1 - z_2|^2 + |\eta_1| + |\eta_2|]^{-1}$, this factor was affordable in [13] since there we considered the regime $|z_1 - z_2| \geq n^{-\epsilon}$ but it would be badly not affordable in the current situation $|z_1 - z_2| \geq n^{-1/2+\epsilon}$. The removal of the factor $\|\mathcal{B}_{12}^{-1}\|$, hence getting an improved bound for any $|z_1 - z_2| \geq n^{-1/2+\epsilon}$, is the main achievement of the new characteristic flow technique. In connection with this improvement, the range of η has also been improved: the estimate in (3.10) holds uniformly in $\eta_* \geq n^{-1+\epsilon}$ while the local law [13, Theorem 5.2] holds only for $\eta_* \geq n^{-1+\epsilon}|z_1 - z_2|^{-2}$, which in the current mesoscopic case can be basically order one, rendering the estimate useless. Furthermore, the error term in (3.10) is better than the one in [13, Theorem 5.2] even in the almost macroscopic regime $|z_1 - z_2| \geq n^{-\epsilon}$ for certain values of η_* . Finally, for a complete comparison we also mention that [13, Theorem 5.2] was valid for all z values, including the edge regime $|z| \approx 1$ of the circular law, while for technical convenience we restricted Theorem 3.3 to the bulk regime $|z| \leq 1 - \tau$.

We stated only the averaged (tracial) version of the two-resolvent local law since this is needed in the proof of our main result Theorem 2.1. However our method would also give an isotropic local law for the matrix elements $\langle \mathbf{x}, G_1 A G_2 \mathbf{y} \rangle$ for deterministic vectors \mathbf{x}, \mathbf{y} with a similar gain from the regime $|z_1 - z_2| \gg \eta^*$ as in (3.10).

By Theorem 3.3 we readily conclude (see Appendices A.1–A.2) the following two propositions, whose combination will prove Theorem 2.1 in Section 4.

Proposition 3.4 (CLT for resolvents). *Let $\epsilon, \xi, \tau, \omega_p, \omega_d > 0$ be small constants and $p \in \mathbf{N}$. Denote by Π_p the set of pairings⁵ on $[p]$. Then for $z_1, \dots, z_p \in \mathbf{C}$, with $|z_i| \leq 1 - \tau$, $n^{-1/2+\omega_p} \leq |z_1 - z_m| \leq n^{-\omega_d}$, and $\eta_1, \dots, \eta_p \geq n^{-1+\epsilon}$, we have*

$$\begin{aligned} \mathbf{E} \prod_{i \in [p]} \langle G_i - \mathbf{E} G_i \rangle &= \sum_{P \in \Pi_p} \prod_{\{i, j\} \in P} \mathbf{E} \langle G_i - \mathbf{E} G_i \rangle \langle G_j - \mathbf{E} G_j \rangle + \mathcal{O}(\Psi) \\ &= \frac{1}{n^p} \sum_{P \in \Pi_p} \prod_{\{i, j\} \in P} \frac{V_{i, j} + \kappa_4 U_i U_j}{2} + \mathcal{O}(\Psi), \end{aligned} \quad (3.14)$$

where $G_i = G^{z_i}(i\eta_i)$,

$$\Psi := \frac{n^\xi}{(n\eta_*)^{1/2}} \prod_{i \in [p]} \frac{1}{n\eta_i}, \quad (3.15)$$

$\eta_* := \min_i \eta_i$, and $V_{i, j} = V_{i, j}(z_i, z_j, \eta_i, \eta_j)$ and $U_i = U_i(z_i, \eta_i)$ are defined as

$$\begin{aligned} V_{i, j} &:= \frac{1}{2} \partial_{\eta_i} \partial_{\eta_j} \log [1 + (u_i u_j |z_i| |z_j|)^2 - m_i^2 m_j^2 - 2u_i u_j \Re z_i \bar{z}_j], \\ U_i &:= \frac{i}{\sqrt{2}} \partial_{\eta_i} m_i^2, \end{aligned} \quad (3.16)$$

with $m_i = m^{z_i}(i\eta_i)$ and $u_i = u^{z_i}(i\eta_i)$ from (3.4). Finally, $\kappa_4 := \mathbf{E} |\chi|^4 - 2$ is the fourth cumulant of the random variable χ in Assumption 1.

The expectation of G_i appearing in (3.14) has already been identified with sufficiently high precision in [13, Lemma 6.2]:

$$\langle \mathbf{E} G \rangle = \langle M \rangle - \frac{i\kappa_4}{4n} \partial_\eta (m^4) + \mathcal{O}\left(\frac{1}{|1 - |z||n^{3/2}(1 + \eta)} + \frac{1}{|1 - |z||n\eta^2}\right), \quad (3.17)$$

which holds for $\eta \geq n^{-1+\epsilon}$ and $|z| \leq C$, for some constant $C > 0$.

The following proposition states that $\langle G^{z_1} \rangle$ and $\langle G^{z_2} \rangle$ are asymptotically independent as long as $n^{-1/2} \ll |z_1 - z_2| \ll 1$. This complements [13, Proposition 3.5] which proves a similar result in the regime $|z_1 - z_2| \sim 1$.

⁵Note that $\Pi_p = \emptyset$ if p is odd.

Proposition 3.5 (Independence of resolvents with small imaginary part). *Fix $p \in \mathbf{N}$. For any sufficiently small constants $\tau, \omega_p, \omega_d, \delta_0, \delta_1 > 0$, there exists $\omega \gg \delta_0, \delta_1$ such that for any $|z_l| \leq 1 - \tau$, $n^{-1/2+\omega_p} \leq |z_l - z_m| \leq n^{-\omega_d}$, with $l, m \in [p]$, $l \neq m$, it holds*

$$\mathbf{E} \prod_{l=1}^p \langle G^{z_l}(i\eta_l) \rangle = \prod_{l=1}^p \mathbf{E} \langle G^{z_l}(i\eta_l) \rangle + \mathcal{O} \left(\frac{n^{p\delta_0 + \delta_1}}{n^\omega} \right), \quad (3.18)$$

for any $\eta_1, \dots, \eta_p \in [n^{-1-\delta_0}, n^{-1+\delta_1}]$.

Remark 3.6. *We formulated our results for the complex case. We now briefly explain how the above strategy needs to be modified for the real symmetry class, i.e. what should be changed to extend our real macroscopic CLT proof [16] to the entire mesoscopic regime. Apart from some irrelevant technicalities, the main difference between the proofs for the complex and the real cases is that the singular vector overlap bound in Theorem 3.1 is needed not only for the low lying singular values, but practically for all of them. This requires to prove the improved version of the local law (1.8) for $\text{Tr} G^{z_1}(w_1)G^{z_2}(w_2)$ not only on the imaginary axis, $w_j = i\eta_j$, but for any spectral parameters w_1, w_2 (see [16, Theorem 3.5]). The GFT argument can easily be adjusted to this more general case, but the characteristics flow requires very precise explicit calculations that were simpler on the imaginary axis. These calculations are elementary but fairly tedious, so we omit them here.*

4. MESOSCOPIC CLT FOR LINEAR STATISTICS: PROOF OF THEOREM 2.1

The proof of Theorem 2.1 follows similarly to [13, Section 4] once Propositions 3.4–3.5 are given as inputs. We thus now only present the necessary major steps in the proof of Theorem 2.1 without presenting the detailed (fairly elementary) computations. The computations to get the expectation in (2.5) are completely analogous (indeed easier since here $|z| \leq 1 - \tau$) to [13, Section 4.2], and so omitted. Here we only focus on the computation of the variance and higher moments. We remark that in [13] we assumed that $f \in H_0^{2+\delta}(\Omega)$ instead of $f \in H_0^2(\Omega)$ only to compute the boundary term from the last two lines of [13, Eq. (4.29)]. Since we now have $|z_0| \leq 1 - \tau$ this additional $(+\delta)$ regularity assumption is not needed in this case (see also the related Remark 2.5).

First of all we notice that the main contribution to (3.1) comes when $\eta \gg n^{-1}$, i.e. the regime $I_{\eta_c}^T$ is the only term in (3.1) giving an order one contribution to the linear statistics $L_n(f_{z_0, a})$. This fact is stated in the following lemma whose proof is postponed to the end of this section.

Lemma 4.1. *Fix $\tau > 0$, $|z_0| \leq 1 - \tau$ and $a \in (0, \frac{1}{2})$, then for any $p \in \mathbf{N}$, and for $f_{z_0, a}^{(i)}(z) := f^{(i)}(n^a(z - z_0))$, with $f^{(i)} \in H_0^2(\Omega)$, it holds*

$$\mathbf{E} \prod_{i=1}^p L_n(f_{z_0, a}^{(i)}) = \mathbf{E} \prod_{i=1}^p I_{\eta_c}^T(f_{z_0, a}^{(i)}) + \mathcal{O}(n^{-c(p)}), \quad (4.1)$$

for some small $c(p) > 0$. The implicit constant in $\mathcal{O}(\cdot)$ may depend on p , $\|\Delta f^{(i)}\|_{L^2(\Omega)}$, and $|\Omega|$.

Using Lemma 4.1 we then compute the deterministic approximation of the moments of $L_n(f_{z_0, a})$ via the moments of the leading term $I_{\eta_c}^T$. The proof of this lemma is presented after we conclude the proof of our main result Theorem 2.1.

Lemma 4.2. *Consider $f_{z_0, a}^{(i)}$ as above, and recall that Π_p denotes the set of pairings on $[p]$. Then it holds*

$$\mathbf{E} \prod_{i=1}^p L_n(f_{z_0, a}^{(i)}) = \sum_{P \in \Pi_p} \prod_{\{i, j\} \in P} \left[- \int_{\mathbf{C}} d^2 z_i \Delta f_{z_0, a}^{(i)}(z_i) \int_{\mathbf{C}} d^2 z_j \Delta f_{z_0, a}^{(j)}(z_j) \int_0^\infty d\eta_i \int_0^\infty d\eta_j \frac{V_{i, j} + \kappa_4 U_i U_j}{8\pi^2} \right] + \mathcal{O}(n^{-c(p)}), \quad (4.2)$$

for some small $c(p) > 0$, with the (z_i, η_i) -dependent quantities $V_{i, j}$ and U_i being defined in (3.16). The implicit constant in $\mathcal{O}(\cdot)$ may depend on p , $\|\Delta f^{(i)}\|_{L^2(\Omega)}$, and $|\Omega|$.

We are now ready to prove Theorem 2.1.

Proof of Theorem 2.1. To keep the presentation concise we only present the computations for higher moments, the computations for the expectation are completely analogous (actually easier) and so omitted (see e.g. [13, Section 4.2]).

By Lemma 4.2 we are only left with the computation of the deterministic term in the right hand side of (4.2) for $f^{(1)}, \dots, f^{(p)} \in \{f, \bar{f}\}$. In particular, using the explicit formulas for $V_{i, j}$ and $U_i U_j$ from (3.16), and using the explicit

formulas for m_i, u_i at $\eta_i = 0$, we readily conclude

$$-\int_0^\infty d\eta_i \int_0^\infty d\eta_j [V_{i,j} + \kappa_4 U_i U_j] = -\frac{1}{2} \log |z_i - z_j|^2 + \frac{\kappa_4}{2} (1 - |z_i|^2)(1 - |z_j|^2), \quad (4.3)$$

for $|z_i|, |z_j| \leq 1 - \tau$ (see [13, Sections 4.3.1–4.3.3]).

Then, performing integration by parts in the z_i, z_j -variables and using that

$$-\partial_{z_1} \partial_{\bar{z}_2} \log |z_1 - z_2|^2 d^2 z_1 d^2 z_2 = \frac{\pi}{2} \delta(z_1 - z_2)$$

in the sense of distributions, we obtain⁶

$$\begin{aligned} \mathbf{E} L_n(f_{z_0,a}^{(i)}) L_n(f_{z_0,a}^{(j)}) &\approx \frac{1}{4\pi} \int_{\mathbf{C}} \nabla f_{z_0,a}^{(i)}(z) \cdot \nabla f_{z_0,a}^{(j)}(z) d^2 z + \frac{\kappa_4}{\pi^2} \left(\int_{\mathbf{C}} f_{z_0,a}^{(i)}(z) d^2 z \right) \left(\int_{\mathbf{C}} f_{z_0,a}^{(j)}(z) d^2 z \right) \\ &= \frac{1}{4\pi} \int_{\mathbf{C}} \nabla f^{(i)}(z) \cdot \nabla f^{(j)}(z) d^2 z + \mathcal{O}(n^{-4a}), \end{aligned} \quad (4.4)$$

which concludes the proof of Theorem 2.1. \square

We now conclude this section with the proof of Lemma 4.1 and Lemma 4.2.

Proof of Lemma 4.1. This proof is analogous to [13, Lemma 4.3]. We only present a few minor differences for completeness. Proceeding as in the proof of [13, Lemma 4.3], using that $\|\Delta f_{z_i, a_i}^{(i)}\|_{L^q(\Omega)} = \|\Delta f^{(i)}\|_{L^q(\Omega)}$, $q \geq 1$, we conclude the following a priori bounds

$$|J_T| \leq \frac{n^{1+\xi} \|\Delta f\|_{L^1(\Omega)}}{T^2}, \quad |I_0^{\eta_0}| + |I_{\eta_0}^{\eta_c}| + |I_{\eta_c}^T| \leq n^\xi \|\Delta f\|_{L^2(\Omega)} |\Omega|^{1/2} \quad (4.5)$$

with very high probability for each $f = f^{(i)}$, with $J_T = J_T(f_{z_i, a_i}^{(i)})$, and similarly for the other terms. Additionally, by [49, Theorem 3.2] we conclude

$$\mathbf{E} |I_0^{\eta_0}| \leq n^{-\delta'} \|\Delta f\|_{L^2(\Omega)}, \quad (4.6)$$

for some small fixed $\delta' > 0$. Next we will prove that

$$\mathbf{E} |I_{\eta_0}^{\eta_c}|^2 \leq n^{-\delta'} \|\Delta f\|_{L^2(\Omega)}^2. \quad (4.7)$$

Note that combining (4.5)–(4.7) we immediately conclude (4.1).

For the proof of (4.7) we now compute (here we neglect $\log n$ -factors)

$$\begin{aligned} \mathbf{E} |I_{\eta_0}^{\eta_c}|^2 &= \frac{n^2}{4\pi^2} \int \int_{|z_2 - z_1| \geq n^{-a-\delta_*}} d^2 z_1 d^2 z_2 \Delta f_{z_0,a}(z_1) \overline{\Delta f_{z_0,a}(z_2)} \\ &\quad \times \int \int_{\eta_0}^{\eta_c} d\eta_1 d\eta_2 \mathbf{E} \left[\langle G^{z_1}(i\eta_1) - \mathbf{E} G^{z_1}(i\eta_1) \rangle \langle G^{z_2}(i\eta_2) - \mathbf{E} G^{z_2}(i\eta_2) \rangle \right] + \mathcal{O}\left(\frac{n^{2\xi}}{n^{\delta_*}}\right), \end{aligned} \quad (4.8)$$

⁶Here by \approx we mean that the equality holds up to error of size n^{-c} for some small $c > 0$.

where we used that the regime $|z_1 - z_2| < n^{-a-\delta_*}$ is bounded by $n^{2\xi-\delta_*}$:

$$\begin{aligned}
& \frac{n^2}{4\pi^2} \int \int_{|z_2-z_1| < n^{-a-\delta_*}} d^2 z_1 d^2 z_2 \Delta f_{z_0,a}(z_1) \overline{\Delta f_{z_0,a}(z_2)} \\
& \quad \times \int_{\eta_0}^{\eta_c} d\eta_1 d\eta_2 \mathbf{E} \left[\langle G^{z_1}(i\eta_1) - \mathbf{E} G^{z_1}(i\eta_1) \rangle \langle G^{z_2}(i\eta_2) - \mathbf{E} G^{z_2}(i\eta_2) \rangle \right] \\
& \lesssim n^2 \int \int_{|z_2-z_1| < n^{-a-\delta_*}} d^2 z_1 d^2 z_2 |\Delta f_{z_0,a}(z_1)| |\overline{\Delta f_{z_0,a}(z_2)}| \\
& \quad \times \int_{\eta_0}^{\eta_c} d\eta_1 d\eta_2 \mathbf{E} \left[|\langle G^{z_1}(i\eta_1) - \mathbf{E} G^{z_1}(i\eta_1) \rangle|^2 + |\langle G^{z_2}(i\eta_2) - \mathbf{E} G^{z_2}(i\eta_2) \rangle|^2 \right] \\
& \lesssim n^{2\xi} \int d^2 z_1 |\Delta f_{z_0,a}(z_1)| \left(\int_{|z_2-z_1| < n^{-a-\delta_*}} d^2 z_2 \right)^{1/2} \left(\int_{|z_2-z_1| < n^{-a-\delta_*}} d^2 z_2 |\overline{\Delta f_{z_0,a}(z_2)}|^2 \right)^{1/2} \\
& \lesssim n^{2\xi} \|\Delta f\|_{L^2(\Omega)}^2 n^{-a-\delta_*} n^a \lesssim n^{2\xi-\delta_*},
\end{aligned}$$

where we recall that $f_{z_0,a}(z) = f(n^a(z - z_0))$ by (2.3); more precisely, we used the scaling

$$\|\Delta f_{z_0,a}\|_{L^1(\Omega)} = \|\Delta f\|_{L^1(\Omega)}, \quad \|\Delta f_{z_0,a}\|_{L^2(\Omega)} = n^a \|\Delta f\|_{L^2(\Omega)}.$$

We point out that in the second inequality we also used the averaged local law in (3.6). In particular, we remark that the implicit constant in $\mathcal{O}(\cdot)$ in (4.8) depends on $\|\Delta f\|_{L^2(\Omega)}$ even if not written explicitly.

Then, by Proposition 3.5, we conclude that

$$\mathbf{E} \left[\langle G^{z_1}(i\eta_1) - \mathbf{E} G^{z_1}(i\eta_1) \rangle \langle G^{z_2}(i\eta_2) - \mathbf{E} G^{z_2}(i\eta_2) \rangle \right] = \mathcal{O} \left(\frac{n^{c(\delta_0+\delta_1)}}{n^\omega} \right), \quad (4.9)$$

for some $c > 0$ and $\omega \gg \delta_0 + \delta_1$. Plugging (4.9) in (4.8) we conclude the proof of (4.7). \square

Proof of Lemma 4.2. Again, this proof is basically the same as its macroscopic counterpart in [13, Lemma 4.7]. Using the same notation as in [13, Proof of Lemma 4.7], we define

$$\widehat{Z}_i := \bigcup_{j < i} \{z_j : |z_i - z_j| \leq n^{-a-\nu}\}$$

for some small fixed $\nu > 0$. Then, similarly to (4.8), we start removing the regime $|z_i - z_j| < n^{-a-\nu}$ (recall that $\eta_c = n^{-1+\delta_1}$):

$$\mathbf{E} \prod_{i=1}^p L_n \left(f_{z_0,a}^{(i)} \right) = \frac{(-n)^p}{(2\pi i)^p} \prod_{i \in [p]} \int_{\widehat{Z}_i^c} d^2 z_i \Delta f^{(i)}(z_i) \mathbf{E} \prod_{i \in [p]} \int_{\eta_c}^T \langle G^{z_i}(i\eta_i) - \mathbf{E} G^{z_i}(i\eta_i) \rangle d\eta_i + \mathcal{O}(n^{-c}), \quad (4.10)$$

for some small $c > 0$ which depends on p, ξ, δ_1, ν and that may change from line to line. We point out that, before removing $|z_i - z_j| < n^{-a-\nu}$, here we also used Lemma 4.1 to remove the integration regimes $\eta \in [0, \eta_c)$ and $\eta \in (T, \infty)$.

Then, by Proposition 3.4, it readily follows that

$$\mathbf{E} \prod_{i=1}^p L_n \left(f_{z_0,a}^{(i)} \right) = - \prod_{i=1}^p \int_{\widehat{Z}_i^c} d^2 z_i \Delta f_{z_0,a}^{(i)} \sum_{P \in \Pi_p} \prod_{\{i,j\} \in P} \int_{\eta_c}^T d\eta_i d\eta_j \frac{V_{i,j} + \kappa_4 U_i U_j}{8\pi^2} + \mathcal{O}(n^{-c}). \quad (4.11)$$

Using the following bounds

$$|V_{i,j}| \lesssim \frac{(1+\eta_i)^{-2}(1+\eta_j)^{-2}}{|z_1 - z_2|^2 + \eta_i + \eta_j}, \quad |U_i| \lesssim \frac{1}{(1+\eta_i)^3},$$

from [13, Eq. (4.21)], to add back first the removed η -regimes $[0, \eta_c] \cup [T, \infty)$ and then the regime \widehat{Z}_i in (4.11) we conclude (4.2). \square

5. PROOF OF THEOREM 3.3 FOR MATRICES WITH A GAUSSIAN COMPONENT

Consider the Ornstein-Uhlenbeck (OU) flow

$$dX_t = -\frac{1}{2}X_t dt + \frac{d\mathfrak{B}_t}{\sqrt{n}}, \quad (5.1)$$

with \mathfrak{B}_t a matrix whose entries are complex i.i.d. Brownian motions. Let

$$W_t := \begin{pmatrix} 0 & X_t \\ X_t^* & 0 \end{pmatrix},$$

be the Hermitisation of X_t as in (3.2) (for $z = 0$), and similarly define B_t being the Hermitisation of \mathfrak{B}_t . Furthermore, for $i = 1, 2$, let $G_i = G_t^{z_i}(i\eta_i) := (H_t^{z_i} - i\eta_i)^{-1}$ be the resolvent of $H_t^{z_i} := W_t - Z_i$, with $|\eta_i| > 0$ and (recall the definition of F from (3.9))

$$Z_i := \begin{pmatrix} 0 & z_i \\ \bar{z}_i & 0 \end{pmatrix} = \bar{z}_i F + z_i F^*.$$

In particular we let η_i assume both positive and negative values. Note that along the flow (5.1) the first two moments of X_t are preserved, hence the deterministic approximation of $G_t^{z_i}(i\eta_i)$ is given by $M^{z_i}(i\eta_i)$ for any $t \geq 0$ (see (3.6)), i.e. the deterministic approximation of $G_t^{z_i}(i\eta_i)$ is independent of time.

By (5.1) and Itô's formula, the evolution of $\langle G_t^{z_1} A G_t^{z_2} B \rangle$, for any deterministic matrices A, B , and for time-independent spectral parameters η_i and z_i , is described by the following flow (recall the definition of E_1, E_2 from (3.9)):

$$\begin{aligned} d\langle G_1 A G_2 B \rangle &= \sum_{\alpha} \partial_{\alpha} \langle G_1 A G_2 B \rangle \frac{d(B_t)_{\alpha}}{\sqrt{n}} + \langle G_1 A G_2 B \rangle dt \\ &\quad + 2\langle G_1 A G_2 E_1 \rangle \langle G_2 B G_1 E_2 \rangle dt + 2\langle G_1 A G_2 E_2 \rangle \langle G_2 B G_1 E_1 \rangle dt \\ &\quad + 2\langle G_1 E_1 \rangle \langle G_1 A G_2 B G_1 E_2 \rangle dt + 2\langle G_1 E_2 \rangle \langle G_1 A G_2 B G_1 E_1 \rangle dt \\ &\quad + 2\langle G_2 E_1 \rangle \langle G_2 B G_1 A G_2 E_2 \rangle dt + 2\langle G_2 E_2 \rangle \langle G_2 B G_1 A G_2 E_1 \rangle dt \\ &\quad + \frac{1}{2} \langle (Z_1 + i\eta_1) G_1 A G_2 B G_1 \rangle dt + \frac{1}{2} \langle (Z_2 + i\eta_2) G_2 B G_1 A G_2 \rangle dt. \end{aligned} \quad (5.2)$$

Here $\alpha = (a, b) \in [2n]^2$ denotes a double index, and ∂_{α} denotes the directional derivative $\partial_{w_{\alpha}}$, with $w_{\alpha} = w_{\alpha}(t) := (W_t)_{\alpha}$. We point out that the summation \sum_{α} in (5.2) is restricted to either $a \leq n, n < b \leq 2n$ or $n < a \leq 2n, b \leq n$ even if not stated explicitly; we will use this notation throughout the paper.

We now allow both pairs of spectral parameters, η_i and z_i , $i = 1, 2$, to be time dependent in a specific way. Define

$$\Lambda_i := \begin{pmatrix} i\eta_i & z_i \\ \bar{z}_i & i\eta_i \end{pmatrix},$$

and consider its time evolution along the following differential equation (called the *characteristic equation*)

$$\partial_t \Lambda_{i,t} = -\frac{\Lambda_{i,t}}{2} - \mathcal{S}[M(\Lambda_{i,t})] \quad (5.3)$$

with some initial condition $\Lambda_{i,0}$, with \mathcal{S} being defined in (3.8). Here we used the notation $M(\Lambda_{i,t}) := M^{z_i,t}(i\eta_{i,t})$. Written component-wise, we thus have that

$$\partial_t \eta_{i,t} = -\Im m^{z_i,t}(i\eta_{i,t}) - \frac{\eta_{i,t}}{2}, \quad \partial_t z_{i,t} = -\frac{z_{i,t}}{2}. \quad (5.4)$$

Since $\Im m^z(i\eta)$ is undefined for $\eta = 0$, we will always run this flow up to a maximal time

$$T^* = T^*(\Lambda_{i,0}) = T^*(z_{i,0}, \eta_{i,0}) := \sup\{t : \text{sgn } \eta_{i,t} = \text{sgn } \eta_{i,0}\}$$

to guarantee that $\eta_{i,t}$ never crosses the real axis. Define $m_{i,t} := m^{z_i,t}(i\eta_{i,t})$, $u_{i,t} := u_{z_i,t}(i\eta_{i,t})$, and note that their time dependence is particularly simple:

$$m_{i,t} = e^{t/2} m_{i,0}, \quad u_{i,t} = e^t u_{i,0}, \quad z_{i,t} = e^{-t/2} z_{i,0}, \quad \eta_{i,t} = e^{-t/2} \eta_{i,0} - (e^{t/2} - e^{-t/2}) \Im m_{i,0}. \quad (5.5)$$

⁷Here B denotes a deterministic matrix while B_t denotes a matrix whose entries are i.i.d. Brownian motions; we apologize for this slight abuse of notation.

Additionally, by (3.7) we have $\Im m_{i,0}^{z_{i,0}} = \text{sgn}(\eta_{i,0})\sqrt{1 - |z_{i,0}|^2} + \mathcal{O}(|\eta_{i,0}|)$, which, together with (5.5), for $t \ll 1$ gives $\eta_{i,t} = \eta_{i,0} - \text{sgn}(\eta_{i,0})c_i t$, with some time dependent positive coefficient

$$c_i = c_i(t) = \sqrt{1 - |z_{i,t}|^2} + \mathcal{O}(\eta_{i,0}). \quad (5.6)$$

Note that c_i is well separated from zero along the whole flow, as a consequence of the fact that if initially $|z_{i,0}| \leq 1 - \tau$ for some $\tau > 0$ then we also have $|z_{i,t}| \leq 1 - \tau$ for any $t \geq 0$. In particular, this shows that $|\eta_{i,t}| = |\eta_{i,0}| - c_i t$, so the flow approaches the real axis with a speed of order one in the regime away from the non-Hermitian spectral edge $|z| = 1$. This shows that the characteristics are monotone in time, i.e. $|\eta_{i,t}| \leq |\eta_{i,s}|$ for $s \leq t$.

Define $G_{i,t} := (W_t - \Lambda_{i,t})^{-1}$, then combining (5.2) with (5.3) and using that $Z_i + i\eta_i = \Lambda_i$, we get (in the r.h.s. we use the notation $G_i = G_{i,t}$ for simplicity):

$$\begin{aligned} d\langle G_{1,t} A G_{2,t} B \rangle &= \sum_{\alpha} \partial_{\alpha} \langle G_1 A G_2 B \rangle \frac{dB_{\alpha}}{\sqrt{n}} + \langle G_1 A G_2 B \rangle dt \\ &\quad + 2\langle G_1 A G_2 E_1 \rangle \langle G_2 B G_1 E_2 \rangle dt + 2\langle G_1 A G_2 E_2 \rangle \langle G_2 B G_1 E_1 \rangle dt \\ &\quad + 2\langle (G_1 - M_1) E_1 \rangle \langle G_1 A G_2 B G_1 E_2 \rangle dt \\ &\quad + 2\langle (G_1 - M_1) E_2 \rangle \langle G_1 A G_2 B G_1 E_1 \rangle dt \\ &\quad + 2\langle (G_2 - M_2) E_1 \rangle \langle G_2 B G_1 A G_2 E_2 \rangle dt \\ &\quad + 2\langle (G_2 - M_2) E_2 \rangle \langle G_2 B G_1 A G_2 E_1 \rangle dt. \end{aligned} \quad (5.7)$$

Note that the careful choice of the characteristic ODE (5.3) guarantees that the last line of (5.2) and the leading terms of the third and fourth lines of (5.2) cancel.

Remark 5.1. *Beside the spectral parameter η_t our characteristic flow also moves the additional parameter z_t ; previous applications of the flow method operated only with moving the spectral parameter. The main cancellation concerns the $\langle i\eta G B G A G \rangle$ term in (5.2) and this could be achieved by the correct choice of the η_t -flow alone. However, our choice of time dependent z_t also cancels the $\langle Z G B G A G \rangle$ term automatically, saving us from additional work to estimate it using the off-diagonality of Z . The canonical form (5.3) of the flow seems the most efficient, and it also gives a hint how to find the best characteristic flow for much more general ensembles.*

The main result of this section is the following Proposition 5.3 below that shows how a two-resolvent local law at large η can be propagated to smaller η . In Part 1 we formulate a general estimate, which will then be improved in Part 2 for the special case when $|z_1 - z_2|$ is relatively big compared with η . Note that both results are conditional: given a (small) η where we want to prove the local law, we construct a (larger) η_0 such that after time T the characteristic flow (5.4) with initial condition η_0 ends up precisely at our target $\eta = \eta_T$. Assuming the local law at η_0 , this proposition proves the local law at $\eta = \eta_T$.

Before stating Proposition 5.3, in the following lemma we prove a simple property of the characteristics in (5.4). Recall that $T^* = T^*(z, \eta)$ is the maximal time so that the η_t -flow with initial condition z, η does not cross the real axis.

Lemma 5.2. *Fix n -independent $\tau, \omega_1 > 0$, and pick any $|\eta| > 0$, $0 < T \leq n^{-\omega_1}$, $|z| \leq 1 - \tau$. Then, there exist an initial data η_0, z_0 , with $T^*(z_0, \eta_0) \geq T$, such that the solutions to (5.4) with these initial data satisfy $\eta_T = \eta$, $z_T = z$ after time T . We have $|\eta_0| \gtrsim T$ and $|z_0| \leq 1 - \tau/2$.*

Proof. This lemma is a simple consequence of the fact that the flow $t \rightarrow \eta_{i,t}$, given in the last equation of (5.5), moves toward the real axis with a linear speed that is well separated from zero. To establish this fact, we use that, since the time $T \ll 1$ is short, the right hand side of the second equation of (5.4) is bounded and initially $|z_0|$ is well separated from 1, we see that $|z_t| \leq 1 - \tau/2$ for all $t \leq T \leq n^{-\omega_1}$, in particular we stay in the bulk regime for all $t \in [0, T]$. This guarantees that $|\Im m^z| \geq c$, with some small n -independent constant $c > 0$ along the whole solution up to time T . Thus the right hand side of the first equation in (5.4) is negative, well separated away from zero for all $t \in [0, T]$. This establishes the linear speed of $\eta_{i,t}$. Therefore, if we are given some η, z , and T , with $T \ll 1$ and z in the bulk of the spectrum of X , then by running this approximately linear flow backward in time we can find initial values z_0, η_0 as required in the lemma. \square

Proposition 5.3. *Fix small n -independent constants $\epsilon, \tau, \omega_d > 0$, and, for $i = 1, 2$, let*

$$\Lambda_{i,0} = \begin{pmatrix} i\eta_{i,0} & z_{i,0} \\ \bar{z}_{i,0} & i\eta_{i,0} \end{pmatrix}$$

with $|z_{i,0}| \leq 1 - \tau$, $|z_{1,0} - z_{2,0}| \leq n^{-\omega_d}$ and $\eta_{i,0} \neq 0$. Let $\Lambda_{i,t}$ be the solution of (5.3) with initial condition $\Lambda_{i,0}$ for any $t \leq T^*(z_{i,0}, \eta_{i,0})$. Set $G_{i,t} := (W_t - \Lambda_{i,t})^{-1}$, and denote the deterministic approximation of $G_{1,t}AG_{2,t}$ by $M_{12,t}^A := M_{12}^A(z_{1,t}, i\eta_{1,t}, z_{2,t}, i\eta_{2,t})$ as given in (3.11). Then we have the following statements:

Part 1. Choose $\eta_{i,0}$ such that $|\eta_{i,0}| \leq n^{-\omega_1}$ for some $\omega_1 > 0$. Assume that for any arbitrary small $\xi > 0$ it holds

$$|\langle G_{1,0}AG_{2,0}B - M_{12,0}^AB \rangle| \leq \frac{n^\xi}{n\eta_{*,0}\sqrt{|\eta_{1,0}\eta_{2,0}|}} \quad (5.8)$$

with very high probability, uniformly for matrices A, B with $\|A\| + \|B\| \leq 1$. Then

$$|\langle G_{1,T}AG_{2,T}B - M_{12,T}^AB \rangle| \leq \frac{n^{2\xi}}{n\eta_{*,T}\sqrt{|\eta_{1,T}\eta_{2,T}|}}, \quad (5.9)$$

with very high probability, uniformly in $T \leq \min_i T^*(z_{i,0}, \eta_{i,0})$ such that $\eta_{*,T} := \min\{|\eta_{1,T}|, |\eta_{2,T}|\} \geq n^{-1+\epsilon}$ and uniformly in matrices A, B with $\|A\| + \|B\| \leq 1$.

Part 2. Choose $\eta_{i,0}$ such that $|\eta_{i,0}| \lesssim |z_{1,0} - z_{2,0}|^2$. Assume that for any arbitrary small $\xi > 0$ it holds

$$|\langle G_{1,0}AG_{2,0}B - M_{12,0}^AB \rangle| \leq n^\xi \mathcal{E}(n, \eta_{1,0}, \eta_{2,0}), \quad (5.10)$$

with very high probability uniformly in matrices A, B with $\|A\| + \|B\| \leq 1$, for some given error function $\mathcal{E}(n, \eta_{1,0}, \eta_{2,0}) \lesssim (n\eta_{*,0}^2)^{-1}$. Then we have

$$|\langle G_{1,T}AG_{2,T}B - M_{12,T}^AB \rangle| \leq \frac{n^{2\xi}}{n\sqrt{|\eta_{1,T}\eta_{2,T}|\eta_{*,T}\eta_T^* + |z_{1,T} - z_{2,T}|^2}} + \frac{n^{3\xi}}{(n\eta_{*,T})^{3/2}\sqrt{|\eta_{1,T}\eta_{2,T}|}} + n^{2\xi} \mathcal{E}(n, \eta_{1,0}, \eta_{2,0}), \quad (5.11)$$

with very high probability uniformly in $T \leq \min_i T^*(z_{i,0}, \eta_{i,0})$ such that $\eta_{*,T} \geq n^{-1+\epsilon}$ and uniformly in matrices A, B with $\|A\| + \|B\| \leq 1$. Here $\eta_T^* := \max\{|\eta_{1,T}|, |\eta_{2,T}|\}$.

Note that the difference between (5.9) and (5.11) lies in the fact that in (5.11) the leading error term is smaller. However, this bound is a genuine improvement compared to (5.9) only for $\eta_T^* \lesssim |z_{1,T} - z_{2,T}|^2$. The bounds (5.9), (5.11) agree for $\eta_T^* \gtrsim |z_{1,T} - z_{2,T}|^2$ and $|\eta_{1,T}| \sim |\eta_{2,T}|$.

Proof of Proposition 5.3. At the beginning the proofs of both parts will be presented together and then we will specialize to the two cases later. In the sequel we often omit the t -dependence and use $G_i := G_{i,t} = (W_t - \Lambda_{i,t})^{-1}$, $M_i := M^{z_{i,t}}(i\eta_{i,t})$, and a similar definition for m_i, u_i . Using the Schwarz inequality and the Ward identity $GG^* = \Im G/\eta$ we have

$$\begin{aligned} |\langle G_1AG_2BG_1E_2 \rangle| &\leq \langle G_1AG_2G_2^*A^*G_1^* \rangle^{1/2} \langle BG_1E_2E_2^*G_1^*B^* \rangle^{1/2} \\ &\leq \frac{\langle \Im G_1A \Im G_2A^* \rangle^{1/2} \langle \Im G_1B^*B \rangle^{1/2}}{|\eta_{1,t}|\sqrt{|\eta_{2,t}|}} \prec \frac{|\langle \Im G_1A \Im G_2A^* \rangle|^{1/2} \|B\|}{\sqrt{\eta_{*,t}|\eta_{1,t}\eta_{2,t}|}}. \end{aligned} \quad (5.12)$$

We point out that in the second inequality we used that

$$\langle BG_1E_2E_2^*G_1^*B^* \rangle \leq \|E_2\|^2 \langle BG_1G_1^*B^* \rangle \leq \langle BG_1G_1^*B^* \rangle = \frac{\langle \Im G_1B^*B \rangle}{\eta_{1,t}},$$

where the last equality follows by Ward identity, and in the last inequality of (5.12) we used

$$\langle \Im G_1B^*B \rangle \leq \|BB^*\| \langle \Im G_1 \rangle \lesssim \|B\|^2 (\langle \Im M_1 \rangle + |\langle \Im G_1 - \Im M_1 \rangle|) \prec \|B\|^2 + \frac{\|B\|^2}{n|\eta_{1,t}|} \lesssim \|B\|^2,$$

which follows by the imaginary part of the local law $|\langle G_1 - M_1 \rangle| \prec 1/(n|\eta_{1,t}|)$ from (3.6). We remark that the key point in the estimate (5.12) is that the G_1AG_2 block is separated from the rest and the estimate reduces a trace with three G 's to one with two G 's (up to imaginary part), i.e. it is of the similar form as the left hand side in Proposition 5.3. This allows us to have a closed Gronwall-like inequality for products with two resolvents. We point out that this philosophy will be often used within the proof of Proposition 5.3 (see e.g. (5.15) below).

Using the bound (5.12) in (5.7), together with the single resolvent local law (3.6), we get

$$\begin{aligned} d\langle G_{1,t}AG_{2,t}B \rangle &= \sum_{\alpha} \partial_{\alpha} \langle G_1AG_2B \rangle \frac{dB_{\alpha}}{\sqrt{n}} + \langle G_1AG_2B \rangle dt \\ &\quad + \langle \mathcal{S}[G_1AG_2]G_2BG_1 \rangle dt + \mathcal{O} \left(\frac{n^{\xi} |\langle \Im G_1A \Im G_2A^* \rangle|^{1/2}}{n\eta_{*,t}^{3/2} \sqrt{|\eta_{1,t}\eta_{2,t}|}} \right) dt. \end{aligned} \quad (5.13)$$

Here we estimated the error in terms of $|\langle \Im G_1A \Im G_2A^* \rangle|^{1/2}$ and ignored $\|B\|$ for brevity (recall that $\|A\| + \|B\| \leq 1$). Note that to go from (5.7) to (5.13) we also used that from (3.8) we have

$$2\langle G_1AG_2E_1 \rangle \langle G_2BG_1E_2 \rangle + 2\langle G_1AG_2E_2 \rangle \langle G_2BG_1E_1 \rangle = \langle \mathcal{S}[G_1AG_2]G_2BG_1 \rangle.$$

We now consider the stochastic term in (5.13). We first estimate its quadratic variation and then use the Burkholder–Davis–Gundy (BDG) inequality to conclude a bound with very high probability. Let \mathcal{F}_t denote the filtration generated by $(B_s)_{0 \leq s \leq t}$, with B_t from (5.1). The quadratic variation process of $n^{-1/2} \sum_{\alpha} \partial_{\alpha} \langle G_1AG_2B \rangle dB_{\alpha}$ is given by

$$\begin{aligned} &\mathbf{E} \left[\frac{1}{n} \sum_{\alpha, \beta} [\langle G_1 \Delta^{\alpha} G_1 A G_2 B \rangle + \langle G_1 A G_2 \Delta^{\alpha} G_2 B \rangle] \cdot [\overline{\langle G_1 \Delta^{\beta} G_1 A G_2 B \rangle} + \overline{\langle G_1 A G_2 \Delta^{\beta} G_2 B \rangle}] dB_{\alpha} \overline{dB_{\beta}} \Big| \mathcal{F}_t \right] \\ &= \frac{1}{n^2} \sum_{ab} [(\langle G_1 A G_2 B G_1 \rangle)_{ab} + (\langle G_2 B G_1 A G_2 \rangle)_{ab}] \cdot [\overline{(\langle G_1 A G_2 B G_1 \rangle)_{ab}} + \overline{(\langle G_2 B G_1 A G_2 \rangle)_{ab}}] dt \\ &= \frac{1}{n^2} \left[\langle G_1 A G_2 B G_1 E_i G_1^* B^* G_2^* A^* G_1^* E_j \rangle + \langle G_2 B G_1 A G_2 E_i G_2^* A^* G_1^* B^* G_2^* E_j \rangle + 2\Re \langle G_1 A G_2 B G_1 E_i G_2^* A^* G_1^* B^* G_2^* E_j \rangle \right] dt, \end{aligned} \quad (5.14)$$

where α, β denote index pairs, and $(\Delta^{ab})_{cd} = \delta_{ac} \delta_{bd}$. In the last line of (5.14) the indices i, j are summed over two pairs $(i, j) \in \{(1, 2), (2, 1)\}$. Similarly to (5.12), by Schwarz inequality (performed by separating the block G_1AG_2 from the rest), it is easy to see that the quadratic variation is bounded by a multiple of (recall that $\|A\| + \|B\| \leq 1$)

$$\frac{|\langle \Im G_1A \Im G_2A^* \rangle|}{n^2 \eta_{*,t}^2 |\eta_{1,t} \eta_{2,t}|}. \quad (5.15)$$

Here we also used that $\|G_i\| \leq |\eta_{i,t}|^{-1}$ deterministically, and that $|\langle G_i \rangle| \leq 1$ with very high probability by the single resolvent local law (3.6). Then by the BDG inequality we conclude

$$\sup_{0 \leq t \leq T} \left| \int_0^t \sum_{\alpha} \partial_{\alpha} \langle G_1AG_2B \rangle \frac{dB_{\alpha}}{\sqrt{n}} \right| \lesssim n^{\xi} \left(\int_0^T \frac{|\langle \Im G_1A \Im G_2A^* \rangle|}{n^2 \eta_{*,t}^2 |\eta_{1,t} \eta_{2,t}|} dt \right)^{1/2}, \quad (5.16)$$

with very high probability. Recall that $0 < T \ll 1$ is fixed.

Combining (5.13) and (5.16), we get the integral equation

$$\begin{aligned} \langle G_{1,T}AG_{2,T}B \rangle &= \langle G_{1,0}AG_{2,0}B \rangle + \int_0^T \langle \mathcal{S}[G_{1,t}AG_{2,t}]G_{2,t}BG_{1,t} \rangle dt + \int_0^T \langle G_{1,t}AG_{2,t}B \rangle dt \\ &\quad + \mathcal{O} \left(n^{\xi} \left(\int_0^T \frac{|\langle \Im G_{1,t}A \Im G_{2,t}A^* \rangle|}{n^2 \eta_{*,t}^2 |\eta_{1,t} \eta_{2,t}|} dt \right)^{1/2} + n^{\xi} \int_0^T \frac{|\langle \Im G_{1,t}A \Im G_{2,t}A^* \rangle|^{1/2}}{n \eta_{*,t}^{3/2} \sqrt{|\eta_{1,t} \eta_{2,t}|}} dt \right), \end{aligned} \quad (5.17)$$

with very high probability, ignoring the B -error terms. We now start distinguishing the proof of (5.9) and (5.11).

Proof of Part 1. Recall that the *stability operator*, defined as $\mathcal{B}_{12} := 1 - M_1 \mathcal{S}[\cdot] M_2$, with $M_i = M^{z_i}(i\eta_i)$, acts on the Hilbert space of $(2n) \times (2n)$ matrices equipped with the usual Hilbert-Schmidt scalar product. It will play a key role in the analysis, in fact we will need to compute the inverse of its adjoint $[\mathcal{B}_{12}^*]^{-1}$. In Appendix B we collected all precise information on the eigenvalues and left/right eigenvectors⁸ of \mathcal{B}_{12} , giving immediately the spectral data of \mathcal{B}_{12}^* as well. First, the 2×2 block structure of M_i and \mathcal{S} shows that \mathcal{B}_{12} and \mathcal{B}_{12}^* are just the identity on the $4n^2 - 4$ dimensional subspace of block traceless matrices. So effectively we need to understand \mathcal{B}_{12} on the four dimensional subspace of block

⁸These are actually matrices, but we will call them eigenvectors since we view them as elements of the vector space of $(2n) \times (2n)$ matrices.

constant matrices that is invariant for both \mathcal{B}_{12} and \mathcal{B}_{12}^* . The main point is that in our relevant regime, $|\eta_i| \leq n^{-\omega_2}$, $|z_i| \leq 1 - \tau$ and $|z_1 - z_2| \leq n^{-\omega_d}$, only one simple eigenvalue, denoted by β_- , is very small, a second eigenvalue, denoted by β_+ is well separated away from zero,

$$|\beta_-| \sim |z_1 - z_2|^2 + \eta_1 + \eta_2, \quad |\beta_+| \sim 1, \quad (5.18)$$

see (B.3), and all the other eigenvalues are 1 with multiplicity $4n^2 - 2$. The left and right eigenvectors corresponding to β_{\pm} , denoted by $L_{\pm} = L_{12,\pm}$ and $R_{\pm} = R_{12,\pm}$, are block constant matrices, defined to satisfy⁹

$$\mathcal{B}_{12}[R_{\pm}] = \beta_{\pm} R_{\pm}, \quad \mathcal{B}_{12}^*[L_{\pm}^*] = \overline{\beta_{\pm}} L_{\pm}^*, \quad (5.19)$$

noting that the two nontrivial eigenvalues of \mathcal{B}_{12}^* are $\overline{\beta_{\pm}}$ (here we dropped the 12 indices). Explicit formulas are given in Appendix B but they are largely irrelevant for us, here we only remark that it is possible to choose the normalization such that $\|L_{\pm}\| \sim 1$, $\|R_{\pm}\| \sim 1$ and $|\langle R_-^* L_-^* \rangle| \sim 1$. In fact, the spectral data corresponding to β_+ , as well as eigenvectors corresponding to the trivial eigenvalues 1 will not be necessary for our main argument. Also notice that the eigenvalues of \mathcal{B}_{12} and \mathcal{B}_{21} are the same (see (B.1) in Appendix B), but their eigenvectors are not identical, e.g. $R_{12,\pm} \neq R_{21,\pm}$.

The inverse of \mathcal{B}_{12}^* can be computed by its spectral decomposition, separating the one dimensional (non-orthogonal) spectral projection Π_- corresponding to the small eigenvalue β_- from the spectral projection Π corresponding to all the other eigenvalues. Explicitly, any matrix Q can be decomposed as

$$Q = \Pi_- [Q] + \Pi [Q], \quad \Pi_- [Q] := \frac{\langle R_-^* Q \rangle}{\langle R_-^* L_-^* \rangle} L_-^*, \quad \Pi [Q] := \frac{1}{2\pi i} \oint \frac{1}{z - \mathcal{B}_{12}^*} [Q] dz, \quad (5.20)$$

with the integral over a contour which encircles $\overline{\beta_+}$ and 1, excludes $\overline{\beta_-}$ and is well separated (order one away) from all eigenvalues. The resolvent $(z - \mathcal{B}_{12}^*)^{-1}$ can be viewed only on the four dimensional invariant subspace of block constant matrices, hence its norm is bounded as all four eigenvalues of $z - \mathcal{B}_{12}^*$ are well separated away from zero when z is on the contour. Thus both Π and Π_- are bounded;

$$\|\Pi\| + \|\Pi_-\| \lesssim 1. \quad (5.21)$$

We clearly have that $(\mathcal{B}_{12}^*)^{-1}$ is bounded on the range of Π , i.e. for any matrix Q

$$\|(\mathcal{B}_{12}^*)^{-1}[\Pi[Q]]\| \lesssim \|Q\|. \quad (5.22)$$

Since $(\mathcal{B}_{12}^*)^{-1}$ on the complementary one dimensional spectral subspace $\text{Span}(L_-^*)$ can be very large, of order $|\beta_-|^{-1}$, this subspace requires a separate treatment. Note that all these bounds trivially extend to $(\mathcal{B}_{12}^*)^{-1}$ viewed on the space of all $(2n) \times (2n)$ matrices due to the invariance of the space of block traceless matrices.

Owing to the time evolution, we actually need the spectral data for $\mathcal{B}_{12,t}[\cdot] := (1 - M_{1,t} \mathcal{S}[\cdot] M_{2,t})$ for small times $t \geq 0$. Let $L_{\pm,t}$ and $R_{\pm,t}$ be the left and right eigenvectors of $\mathcal{B}_{12,t}$ with corresponding eigenvalues $\beta_{\pm,t}$, then by (5.5) it readily follows that $L_{\pm,t} = L_{\pm,0}$, $R_{\pm,t} = e^t R_{\pm,0}$ for any $t \geq 0$. Since the difference between the eigenvectors at time 0 and time t amounts to a simple rescaling by an irrelevant factor $e^t = 1 + \mathcal{O}(t)$, we can use the zero-time eigenvectors $L_{\pm} := L_{\pm,0}$, $R_{\pm} := R_{\pm,0}$ for all later times. The eigenvalues $\beta_{\pm,t}$ depend on t nontrivially but smoothly and the time dependent version of (5.18) holds, in particular $\beta_{-,t}$ still remains well separated from the rest of the spectrum if $t \ll 1$.

After all these preparations, we first handle (5.9) in the case when either A^* or B^* lies in the range of Π . The proof relies on the following technical lemma (whose proof is postponed to the Appendix A):

Lemma 5.4. *Fix any small $\tau, \epsilon > 0$, and fix z_i, η_i , with $i = 1, 2$, such that $|z_i| \leq 1 - \tau$, $|\eta_i| \geq n^{-1+\epsilon}$. Let A, B be any deterministic matrices with $\|A\| + \|B\| \leq 1$ and such that at least one among $\|[\mathcal{B}_{21}^{-1}]^*[A^*]\|$, $\|[\mathcal{B}_{12}^{-1}]^*[B^*]\|$ is bounded by an (n, η) -independent constant, where $\mathcal{B}_{12}[\cdot] := 1 - M_1 \mathcal{S}[\cdot] M_2$, $M_i = M^{z_i}(i\eta_i)$. Then it holds*

$$|\langle (G_1 A G_2 - M_{12}^A) B \rangle| \prec \frac{1}{n\eta_* \sqrt{|\eta_1 \eta_2|}}, \quad (5.23)$$

uniformly in $\eta_* \geq n^{-1+\epsilon}$.

⁹Here we deliberately use the convention that a left eigenvector L is defined such that L^* is the right eigenvector of the adjoint operator. This convention will simplify many formulas below.

Given a deterministic matrix A , we split A^* as in (5.20), then, by (5.22), it clearly follows that $\|(\mathcal{B}_{12}^{-1})^*[\Pi[A^*]]\|$ is bounded. In particular, by (5.20) and bilinearity of (5.9), we obtain that Lemma 5.4 proves (5.9) in all cases except for when A^* and B^* are in the range of the corresponding Π_- . Thus the remainder of the proof focuses on the case $A = L_{21,-}$ and $B = L_{12,-}$, where we recall from (5.19) that $L_{12,-}$ is defined by $\mathcal{B}_{12}^*[L_{12,-}^*] = \overline{\beta_-} L_{12,-}^*$. From now on we introduce the shorthand notations

$$L_{\pm} := L_{12,\pm}, \quad L'_{\pm} := L_{21,\pm}.$$

For definiteness we only consider the case when $\eta_{1,t}\eta_{2,t} < 0$. In this case we have (see (B.6)):

$$L_- = (1 + \mathcal{O}(|z_{1,t} - z_{2,t}|))I + \mathcal{O}(|z_{1,t} - z_{2,t}|)E_-,$$

and an analogous relation for L'_- . The case $\eta_{1,t}\eta_{2,t} > 0$ is completely analogous and so omitted.

Define the stopping time

$$\tau_1 := \inf \left\{ t \geq 0 : \left| \langle (G_{1,t}(i\eta_{1,t})L'_- G_{2,t}(i\eta_{2,t}) - M_{12,t}^{L'_-})L_- \rangle \right| = \frac{n^{2\xi}}{n\eta_{*,t}\sqrt{|\eta_{1,t}\eta_{2,t}|}} \right\} \wedge T. \quad (5.24)$$

Here $\xi \leq (\epsilon \wedge \omega_d)/10$ with $\epsilon > 0$ such that $\eta_{*,t} \geq n^{-1+\epsilon}$ and ω_d such that $|z_{1,t} - z_{2,t}| \lesssim n^{-\omega_d}$ for any $t \geq 0$. We remark that L_- , L'_- in (5.24) are independent of time.

Define

$$Y_t := \langle (G_{1,t}(i\eta_{1,t})L'_- G_{2,t}(i\eta_{2,t}) - M_{12,t}^{L'_-})L_- \rangle.$$

In order to study the time evolution of Y_t , we need to understand how $M_{12,t}^{L'_-}$ evolves in time. This is explained in the following lemma, whose proof is postponed to Appendix A (see (A.26)).

Lemma 5.5. *For any $A, B \in \mathbb{C}^{2n \times 2n}$ It holds that*

$$\partial_t \langle M_{12,t}^A B \rangle = \langle M_{12,t}^A B \rangle + \langle \mathcal{S}[M_{12,t}^A] M_{21,t}^B \rangle. \quad (5.25)$$

Then, choosing $A = L'_-$, $B = L_-$ in (5.17) and (5.25), we obtain

$$\begin{aligned} Y_T &= Y_0 + \int_0^T Y_t dt + \int_0^T [\langle \mathcal{S}[G_{1,t}L'_- G_{2,t}]G_{2,t}L_- G_{1,t} \rangle - \langle \mathcal{S}[M_{12,t}^{L'_-}]M_{21,t}^{L_-} \rangle] dt \\ &+ \mathcal{O} \left(n^\xi \left(\int_0^T \frac{|\langle \Im G_{1,t}L'_- \Im G_{2,t}(L'_-)^* \rangle|}{n^2 \eta_{*,t}^2 |\eta_{1,t}\eta_{2,t}|} dt \right)^{1/2} + n^\xi \int_0^T \frac{|\langle \Im G_{1,t}L'_- \Im G_{2,t}(L'_-)^* \rangle|^{1/2}}{n\eta_{*,t}^{3/2} \sqrt{|\eta_{1,t}\eta_{2,t}|}} dt \right). \end{aligned} \quad (5.26)$$

Next, to estimate the last term in the first line of (5.26) we rely on the following lemma, whose proof is postponed to Appendix A.

Lemma 5.6. *Denote $M_{12,t}^{L'_-} = M_{12,t'}^{L'_-}$, $G_i = G_{i,t}(i\eta_{i,t})$, then it holds*

$$\langle \mathcal{S}[G_1 L'_- G_2] G_2 L_- G_1 \rangle = \langle \mathcal{S}[M_{12,t}^{L'_-}] M_{21,t}^{L_-} \rangle + 2 \langle M_{12,t}^I \rangle Y_t + \mathcal{O} \left((1 + |Y_t|) |Y_t| + \frac{n^\xi}{n\eta_{*,t}^2 \sqrt{|\eta_{1,t}\eta_{2,t}|}} \right). \quad (5.27)$$

Combining (5.26) with (5.27), we conclude

$$Y_t = Y_0 + 2 \int_0^t \langle M_{12,s}^I \rangle Y_s ds + \mathcal{O} \left(\int_0^t (1 + |Y_s|) |Y_s| ds + \frac{n^\xi}{n\eta_{*,t} \sqrt{|\eta_{1,t}\eta_{2,t}|}} \right), \quad (5.28)$$

where we used that $|\langle \Im G_{1,t}L'_- \Im G_{2,t}(L'_-)^* \rangle| \leq \eta_{*,t}^{-1}$ with very high probability, by Schwarz inequality and the single resolvent local law (3.6), to estimate the error terms in the second line of (5.26). Finally, using that, by the definition of the stopping time τ_1 , we have $|Y_s| \leq n^{2\xi} (n\eta_{*,s} \sqrt{|\eta_{1,s}\eta_{2,s}|})^{-1}$, for $0 \leq s \leq \tau_1$, to estimate $(1 + |Y_s|)|Y_s|$ (here we also used that $\xi \leq \epsilon/10$), we conclude

$$Y_t = Y_0 + 2 \int_0^t \langle M_{12,s}^I \rangle Y_s ds + \mathcal{O} \left(\frac{n^\xi}{n\eta_{*,t} \sqrt{|\eta_{1,t}\eta_{2,t}|}} \right). \quad (5.29)$$

We now precisely estimate the deterministic term $\langle M_{12,s}^I \rangle$ in (5.29) (the proof is postponed to Appendix A):

Lemma 5.7. *Let $m_{i,t} := m^{z_{i,t}}(i\eta_{i,t})$, $u_{i,t} := u^{z_{i,t}}(i\eta_{i,t})$, then we have*

$$|\langle M_{12,t}^I \rangle| \leq \frac{a_0}{b_0 - a_0 t} (1 + \mathcal{O}(|\eta_{1,0}| + |\eta_{2,0}| + |z_{1,0} - z_{2,0}|^2 + t)), \quad (5.30)$$

where

$$a_0 := 2 - |z_{1,0}|^2 - |z_{2,0}|^2, \quad b_0 := |z_{1,0} - z_{2,0}|^2 + |\eta_{1,0}| \sqrt{1 - |z_{1,0}|^2} + |\eta_{2,0}| \sqrt{1 - |z_{2,0}|^2}.$$

In order to obtain an estimate for Y_t from (5.29) we will use a Gronwall inequality. For this purpose we compute

$$\exp\left(2 \int_s^t |\langle M_{12,r}^I \rangle| dr\right) \leq \exp\left(2 \int_s^t \left| \frac{a_0}{b_0 - a_0 r} \right| dr\right) = \left(\frac{b_0 - a_0 s}{b_0 - a_0 t}\right)^2 \sim \left(\frac{|z_{1,s} - z_{2,s}|^2 + |\eta_{1,s}| + |\eta_{2,s}|}{|z_{1,t} - z_{2,t}|^2 + |\eta_{1,t}| + |\eta_{2,t}|}\right)^2. \quad (5.31)$$

Here in the last step we used various elementary properties of $z_{i,t}$ and $\eta_{i,t}$ that follow from (5.5). In particular, we used that $\eta_{i,t} = \eta_{i,0} - \text{sgn}(\eta_{i,0})c_i t$, with c_i given in (5.6), and that $1 - |z_{i,t}| \sim 1 - |z_{i,t}| \sim 1 - |z_{i,0}|$, as well as $|z_{1,t} - z_{2,t}| \sim |z_{1,s} - z_{2,s}| \sim |z_{1,0} - z_{2,0}|$, for the times we consider $0 \leq s \leq t \leq T \ll 1$. Combining these information we obtain

$$b_0 - a_0 t = |z_{1,0} - z_{2,0}|^2 + |\eta_{1,0}| \sqrt{1 - |z_{1,0}|^2} + |\eta_{2,0}| \sqrt{1 - |z_{2,0}|^2} - t[2 - |z_{1,0}|^2 - |z_{2,0}|^2] \sim |z_{1,t} - z_{2,t}|^2 + |\eta_{1,t}| + |\eta_{2,t}|,$$

which was used in the last step of (5.31). Note that in our regime $|z_{1,t} - z_{2,t}| \lesssim |\eta_{1,t}| + |\eta_{2,t}|$ for any $0 \leq t \leq T$, hence the $|z_{1,t} - z_{2,t}|$ terms are negligible in (5.31). Using again the elementary properties of $z_{i,t}$ and $\eta_{i,t}$ described above, we note that

$$\frac{|\eta_{1,s}| + |\eta_{2,s}|}{|\eta_{1,t}| + |\eta_{2,t}|} \lesssim 1 + (t-s) \frac{(\sqrt{1 - |z_{1,0}|^2} + \sqrt{1 - |z_{2,0}|^2})}{|\eta_{1,t}| + |\eta_{2,t}|} \lesssim 1 + (t-s) \frac{(\sqrt{1 - |z_{1,0}|^2} + \sqrt{1 - |z_{2,0}|^2})}{\eta_{*,t}} \lesssim \frac{\eta_{*,s}}{\eta_{*,t}}, \quad (5.32)$$

by $1 - |z_{i,t}| \sim 1 - |z_{i,t}| \sim 1 - |z_{i,0}|$ and $|\eta_{i,s}| = |\eta_{i,t}| + (c_i + o(1))(t-s)$, with c_i defined below (5.5). In particular, this also implies that

$$\frac{|\eta_{1,s}| + |\eta_{2,s}|}{|\eta_{1,t}| + |\eta_{2,t}|} \lesssim \frac{\sqrt{|\eta_{1,s}\eta_{2,s}|}}{\sqrt{|\eta_{1,t}\eta_{2,t}|}}. \quad (5.33)$$

Combining (5.31)-(5.33), we conclude

$$\exp\left(2 \int_s^t |\langle M_{12,r}^I \rangle| dr\right) \lesssim \frac{\eta_{*,s} \sqrt{|\eta_{1,s}\eta_{2,s}|}}{\eta_{*,t} \sqrt{|\eta_{1,t}\eta_{2,t}|}}. \quad (5.34)$$

Finally, using (5.34), by a simple Gronwall inequality, together with (5.8) to bound Y_0 , we conclude

$$|Y_t| \lesssim \frac{n^\xi}{n \eta_{*,t} \sqrt{|\eta_{1,t}\eta_{2,t}|}}, \quad (5.35)$$

with very high probability for any $t < \tau_1$. This proves that $\tau_1 = T$ and concludes the proof of (5.9).

Proof of Part 2. Since $|\eta_{i,0}| \lesssim |z_{1,0} - z_{2,0}|^2$ by assumption, and

$$\begin{aligned} \eta_{i,t} &= \eta_{i,0} - (1 - |z_{i,0}|^2)^{1/2} t + \mathcal{O}(t|\eta_{i,0}|), \\ |z_{1,0} - z_{2,0}|^2 &= e^t |z_{1,t} - z_{2,t}|^2 = |z_{1,t} - z_{2,t}|^2 (1 + \mathcal{O}(t)), \end{aligned}$$

we also have that $\eta_{i,t}^* \lesssim |z_{1,t} - z_{2,t}|^2$ for any $0 \leq t \leq T$ in our perturbative regime $T \lesssim |z_{1,0} - z_{2,0}|^2 \leq n^{-2\omega_d}$.

Unlike in Part 1 (see (5.20)), we do not need to separate the spectral projection corresponding to the smallest eigenvalue β_- , this is because the norm estimate (5.21) combined with (5.18) giving

$$\|(\mathcal{B}_{12}^*)^{-1}\| \lesssim |\beta_-|^{-1} \lesssim |z_1 - z_2|^{-2} \quad (5.36)$$

is affordable in the regime of Part 2 when $|z_1 - z_2|$ is relatively large. Therefore we can use a standard orthogonal decomposition in the space of block constant matrices and we use the orthonormal basis $\{I, E_-, \sqrt{2}F, \sqrt{2}F^*\}$ in which the covariance operator \mathcal{S} is particularly simple. We thus decompose any deterministic matrices A as

$$A =: \langle A \rangle I + \langle A E_- \rangle E_- + 2 \langle A F^* \rangle F + 2 \langle A F \rangle F^* + \mathring{A}, \quad (5.37)$$

with \mathring{A} being defined by this formula. In particular, \mathring{A} is just the orthogonal projection (with respect to the usual Hilbert-Schmidt scalar product) of A onto the space of block traceless matrices, i.e. matrices whose all four blocks are traceless. Let $A, B \in \{E_-, I, F, F^*\}$, then we define

$$Y_t := \sup_{A, B \in \{E_-, I, F, F^*\}} |\langle (G_{1,t}(i\eta_{1,t})AG_{2,t}(i\eta_{2,t})B - M_{12,t}^A(i\eta_{1,t}, i\eta_{2,t}))B \rangle| \\ + \sup_{A, B \in \{E_-, I, F, F^*\}} |\langle (G_{1,t}(i\eta_{1,t})AG_{2,t}(-i\eta_{2,t})B - M_{12,t}^A(i\eta_{1,t}, -i\eta_{2,t}))B \rangle|. \quad (5.38)$$

Choose $\xi \leq \epsilon/10$ and define the stopping time

$$\tau_2 := \inf \left\{ t \geq 0 : Y_t = \frac{n^{2\xi}}{n\sqrt{\eta_{*,t}|\eta_{1,t}\eta_{2,t}|(|z_{1,t} - z_{2,t}|^2 + \eta_t^*)}} + \frac{1}{\sqrt{n\eta_{*,t}}} \cdot \frac{n^{3\xi}}{n\eta_{*,t}\sqrt{|\eta_{1,t}\eta_{2,t}|}} + n^{2\xi}\mathcal{E}(n, \eta_{1,0}, \eta_{2,0}) \right\} \wedge T.$$

Note that by the definition of τ_2 and the assumption $|\mathcal{E}(n, \eta_{1,0}, \eta_{2,0})| \lesssim (n\eta_{*,0}^2)^{-1} \lesssim (n\eta_{*,t}^2)^{-1}$ it follows that $Y_t \leq n^{2\xi}(n\eta_{*,t}^2)^{-1}$ for any $t < \tau_2$. Here we also used that $\xi \leq \epsilon/10$ (recall that $\eta_{*,t} \geq n^{-1+\epsilon}$ for any $t \geq 0$) so that $n^\xi/\sqrt{n\eta_{*,t}} \leq 1$

We now proceed similarly to (5.27)–(5.29), we thus do not write all the details but only explain the main differences. By adding and subtracting the deterministic approximation of all the terms in (5.17), by using Lemma 5.5 to show that all the deterministic terms exactly cancel, and that $\|M_{12,t}^A\| \lesssim 1/|z_{1,t} - z_{2,t}|^2$, for any $t \leq \tau_2$, we obtain

$$Y_t \leq Y_0 + C \int_0^t \left(\frac{1}{|z_{1,s} - z_{2,s}|^2} + Y_s \right) Y_s ds + n^\xi \left(\int_0^t \frac{Y_s}{n^2\eta_{*,s}^2|\eta_{1,s}\eta_{2,s}|} ds \right)^{1/2} + n^\xi \int_0^t \frac{Y_s^{1/2}}{n\eta_{*,s}^{3/2}\sqrt{|\eta_{1,s}\eta_{2,s}|}} ds \\ + \frac{n^\xi}{n\sqrt{\eta_{*,t}|\eta_{1,t}\eta_{2,t}|(|z_{1,t} - z_{2,t}|^2 + \eta_t^*)}} \\ \leq Y_0 + C \int_0^t \left(\frac{1}{|z_{1,s} - z_{2,s}|^2} + Y_s + \frac{n^\xi}{\sqrt{n\eta_{*,s}^{3/2}}} \right) Y_s ds + \frac{n^\xi}{N\sqrt{\eta_{*,t}|\eta_{1,t}\eta_{2,t}|(|z_{1,t} - z_{2,t}|^2 + \eta_t^*)}} \\ + \frac{1}{\sqrt{n\eta_{*,t}}} \cdot \frac{n^{2\xi}}{n\eta_{*,t}\sqrt{|\eta_{1,t}\eta_{2,t}|}}, \quad (5.39)$$

for some constant $C > 0$. Note that in the last inequality we first used Schwarz inequality (here we also use the monotonicity of the characteristics $|\eta_{i,t}| \leq |\eta_{i,s}|$ for $s \leq t$)

$$\frac{Y_s^{1/2}}{n\eta_{*,s}^{3/2}\sqrt{|\eta_{1,s}\eta_{2,s}|}} \leq \frac{Y_s}{\sqrt{n\eta_{*,s}^{3/2}}} + \frac{1}{\sqrt{n\eta_{*,s}}} \cdot \frac{1}{N\eta_{*,s}^2\sqrt{|\eta_{1,s}\eta_{2,s}|}} \\ \left(\int_0^t \frac{Y_s}{n^2\eta_{*,s}^2|\eta_{1,s}\eta_{2,s}|} ds \right)^{1/2} \leq \frac{1}{n^{3/4}\eta_{*,t}^{3/4}(\eta_{1,s}\eta_{2,s})^{1/4}} \left(\int_0^t \frac{Y_s}{\sqrt{n\eta_{*,s}^{3/2}}} ds \right)^{1/2} \\ \leq \frac{1}{\sqrt{n\eta_{*,t}}} \cdot \frac{1}{n\eta_{*,t}\sqrt{|\eta_{1,t}\eta_{2,t}|}} + \int_0^t \frac{Y_s}{\sqrt{n\eta_{*,s}^{3/2}}} ds,$$

and then we used that $Y_s \leq n^{2\xi}(n\eta_{*,s}^2)^{-1}$ for $s \leq \tau_2$. Finally, by a simple Gronwall inequality, using that $T \lesssim |z_{1,0} - z_{2,0}|^2$ and the bound $Y_0 \leq n^\xi\mathcal{E}(n, \eta_{1,0}, \eta_{2,0})$ from (5.10), we conclude that

$$Y_t \lesssim \frac{n^\xi}{n\sqrt{\eta_{*,t}|\eta_{1,t}\eta_{2,t}|(|z_{1,t} - z_{2,t}|^2 + \eta_t^*)}} + \frac{1}{\sqrt{n\eta_{*,t}}} \cdot \frac{n^{2\xi}}{n\eta_{*,t}\sqrt{|\eta_{1,t}\eta_{2,t}|}} + n^\xi\mathcal{E}(n, \eta_{1,0}, \eta_{2,0}),$$

for any $t \leq \tau_2$. This shows that $\tau_2 = T$ and thus that (5.11) holds, completing the proof of Proposition 5.3. \square

6. GFT: PROOF OF THEOREM 3.3

The goal of this section is to show that we can remove the Gaussian components added in Section 5. Consider the Ornstein-Uhlenbeck flow

$$dX_t = -\frac{1}{2}X_t dt + \frac{dB_t}{\sqrt{N}}. \quad (6.1)$$

and define

$$R_t := \langle G_1^t A G_2^t B - M_{12}^A B \rangle, \quad (6.2)$$

with

$$G_i^t := (W_t - Z_i - i\eta_i)^{-1}, \quad Z_i := \begin{pmatrix} 0 & z_i \\ \bar{z}_i & 0 \end{pmatrix}, \quad W_t := \begin{pmatrix} 0 & X_t \\ X_t^* & 0 \end{pmatrix}, \quad (6.3)$$

where t in G_i^t denotes time dependence and it should not be confused with the transpose G_i^T . Here we recall that A, B are generic deterministic square matrices of bounded norm. Also we mention that this G_i^t is not the same as $G_{i,t} = (W_t - \Lambda_{i,t})^{-1}$ used in Section 5 since now both spectral parameters z and η are time independent and only W_t changes with time.

Note that along the OU flow the first two moments of X_t are preserved hence M_{12}^A is independent of time. Our main technical result of this section is the following Proposition 6.1.

Proposition 6.1. *Let A, B be arbitrary deterministic matrices with $\|A\| + \|B\| \leq 1$, and let z, η_1, η_2 be spectral parameters with $|z| \leq 1 - \tau$ and $\eta_* := |\eta_1| \wedge |\eta_2| \geq n^{-1+\epsilon}$ for some fixed $\epsilon, \tau > 0$. Then for any $\xi > 0$ and any even $p \geq 4$ it holds that*

$$|\mathbf{d} \mathbf{E}|R_t|^p| \lesssim \sum_{k=4}^p \frac{n^{2-k/2+\xi}}{(n\eta_1\eta_2)^k} \mathbf{E}|R_t|^{p-k} + n^\xi \left(1 + \frac{1}{\sqrt{n}\eta_*}\right) \mathbf{E} \left[\frac{|R_t|^{p-1}}{n\eta_1\eta_2} + |R_t|^{p-3} \left(\frac{1}{n\eta_1\eta_2}\right)^3 \right]. \quad (6.4)$$

From Proposition 6.1 we obtain the following Proposition 6.2 by integration:

Proposition 6.2. *Let X be an i.i.d. matrix, and let X_t the solution of the OU flow (5.1), with initial data $X_0 = X$. Then for any small $\tau, \epsilon > 0$, for any $1 \geq |\eta_i| \geq n^{-1+\epsilon}$, $|z_i| \leq 1 - \tau$, and for any $p \in \mathbf{N}$, denoting $G_i^t := (H_t - Z_i - i\eta_i)^{-1}$, it holds:*

$$\begin{aligned} \left(\mathbf{E} |\langle (G_1^0 A G_2^0 - M_{12}^A) B \rangle|^p \right)^{1/p} &\lesssim e^{t \left(\frac{\eta^* + |z_1 - z_2|^2}{\eta_*} + \frac{1}{\sqrt{n}\eta_*^3/2} \right)} \left[\left(\mathbf{E} |\langle (G_1^t A G_2^t - M_{12}^A) B \rangle|^p \right)^{1/p} \right. \\ &\quad \left. + \frac{n^\xi}{n\eta_*\eta^*} \left(\eta_*^{1/6} + n^{-1/10} + \left(\frac{\eta^*}{\eta^* + |z_1 - z_2|^2} \right)^{1/4} \right) \right], \end{aligned} \quad (6.5)$$

for any $0 \leq t \lesssim 1$ and any small $\xi > 0$, where A, B are deterministic matrices with $\|A\| + \|B\| \leq 1$, and $\eta_* := |\eta_1| \wedge |\eta_2|$, $\eta^* := |\eta_1| \vee |\eta_2|$.

We are now ready to prove Theorem 3.3.

Proof of Theorem 3.3. The proof of this theorem follows by an induction argument, and it is divided into two cases: (i) $\eta^* \geq |z_1 - z_2|^2$, (ii) $\eta^* < |z_1 - z_2|^2$. Since the proofs of (i) and (ii) are analogous we mostly focus on the proof in case (i) and then explain the minor changes for the proof of case (ii).

We start with the first step of the induction. By [13, Theorem 5.2] for any z_1, z_2 it holds

$$\left| \langle (G^{z_1}(i\eta_1) A G^{z_2}(i\eta_2) - M_{12}^A) B \rangle \right| \lesssim \frac{n^{2\xi}}{n\eta_* \sqrt{|\eta_1\eta_2|}}, \quad (6.6)$$

with very high probability for any small $\xi > 0$ uniformly in $|\eta_*| \gtrsim n^{-\xi}$. Fix $|z_1|, |z_2| \leq 1 - \tau$ such that $|z_1 - z_2| \leq n^{-\omega_d}$, and η_1, η_2 such that $\eta_* \geq n^{-1/3}$ and $\eta^* \geq |z_1 - z_2|^2$, and fix $T = n^{-\xi}$, then by Lemma 5.2 there exist $\eta_{1,0}, \eta_{2,0}, z_{1,0}, z_{2,0}$ with $|\eta_{i,0}| \gtrsim n^{-\xi}$, $|z_{i,0}| \leq 1 - \tau/2$, such that for the solution of (5.3) with initial condition $\eta_{i,0}, z_{i,0}$ it holds $\eta_{i,T} = \eta_i$, $z_{i,T} = z_i$. Additionally, since (5.8) for those $\eta_{i,0}, z_{i,0}$ is verified thanks to (6.6), by Part 1 of Proposition 5.3 we conclude that

$$\left| \langle (G_T^{z_1}(i\eta_1) A G_T^{z_2}(i\eta_2) - M_{12,T}^A) B \rangle \right| \lesssim \frac{n^{3\xi}}{n\eta_* \sqrt{|\eta_1\eta_2|}}, \quad (6.7)$$

with very high probability uniformly in $\eta_* \gtrsim n^{-1/3}$ and $\eta^* \geq |z_1 - z_2|^2$. Then, by Proposition 6.2, we readily conclude that

$$\left| \langle (G^{z_1}(i\eta_1)AG^{z_2}(i\eta_2) - M_{12}^A)B \rangle \right| \lesssim \frac{n^{3\xi}}{n\eta_* \sqrt{|\eta_1\eta_2|}}, \quad (6.8)$$

with very high probability uniformly in $\eta_* \gtrsim n^{-1/3}$ and $\eta^* \geq |z_1 - z_2|^2$, i.e. the Gaussian component added in Proposition 5.3 can be completely removed using Proposition 6.2. This concludes the first step of the induction that reduced the lower threshold for η_* from $|\eta_*| \gtrsim n^{-\xi}$ in (6.6) to $|\eta_*| \gtrsim n^{-1/3}$ in (6.8).

For the general induction step define $\eta^{(k)} := n^{-1+(2/3)^k} \vee n^{-1+\epsilon}$, we now show that if (6.8) holds for $\eta_* \geq \eta^{(k)}$ then it holds for $\eta_* \geq \eta^{(k+1)}$ as well. Pick z_1, z_2 as above and η_1, η_2 such that $\eta_* \geq \eta^{(k+1)}$ and $\eta^* \geq |z_1 - z_2|^2$. Choose $T = \eta^{(k)}$ and use the local law (6.8) for $\eta_* \geq \eta^{(k)}$ as an initial input to apply Part 1 of Proposition 5.3 again. With the output of this step, together with an application of Proposition 6.2 for $\eta_* \geq \eta^{(k+1)}$, we conclude that (6.8) holds for $\eta_* \geq \eta^{(k+1)}$. Iterating this procedure $k \sim |\log \epsilon|$ times we conclude (3.10) for $\eta^* \geq |z_1 - z_2|^2$ and $\eta_* \geq n^{-1+\epsilon}$. Note that the apparently accumulating factors of n^ξ at each step are not a problem since the exponent ξ can always be redefined before every iteration step. Since ϵ is given at the beginning, the number of iteration steps is finite, hence ξ needs readjustment only finitely many times.

We now prove Theorem 3.3 for the complementary case $\eta^* < |z_1 - z_2|^2$; as before we proceed by induction. We start describing the first step of the induction. Fix $|z_1|, |z_2| \leq 1 - \tau$ such that $|z_1 - z_2| \leq n^{-\omega_d}$, η_1, η_2 with $|\eta_i| < |z_1 - z_2|^2$, and choose $T = C|z_1 - z_2|^2$ for a constant $C > 0$. Applying Lemma 5.2, there exist $|\eta_{i,0}| \geq |z_1 - z_2|^2$ and $z_{i,0}$ as initial conditions of the characteristics flow so that $\eta_{i,T} = \eta_i$ and $z_{i,T} = z_i$. Additionally, since $T \lesssim |z_{1,0} - z_{2,0}|^2$, by (5.4) it follows that $|\eta_{i,0}| \lesssim |z_{1,0} - z_{2,0}|^2$. Then by an application of Part 2 of Proposition 5.3, with $\mathcal{E}(n, \eta_{1,0}, \eta_{2,0}) = (n\eta_{1,0}\eta_{2,0})^{-1}$ for the $\eta_{i,0}, z_{i,0}$ from Lemma 5.2 and Proposition 6.2 we conclude

$$\begin{aligned} \left| \langle (G^{z_1}(i\eta_1)AG^{z_2}(i\eta_2) - M_{12}^A)B \rangle \right| &\lesssim \frac{n^\xi}{n\eta_*\eta^*} \left(\eta_*^{1/6} + n^{-1/10} + \left(\frac{\eta^*}{\eta^* + |z_1 - z_2|^2} \right)^{1/4} \right) \\ &\quad + \frac{n^{2\xi}}{n\sqrt{|\eta_1\eta_2|\eta_{*,T}(\eta^* + |z_1 - z_2|^2)}} + \frac{n^{3\xi}}{(n\eta_*)^{3/2}\sqrt{|\eta_1\eta_2|}} \\ &\quad + n^{2\xi}\mathcal{E}(n, \eta_{1,0}, \eta_{2,0}) \\ &\lesssim \frac{n^{3\xi}}{n\eta_*^{3/2}(\eta^*)^{1/2}} \left(\eta_*^{1/6} + n^{-1/10} + \frac{1}{\sqrt{n\eta_*}} + \left(\frac{\eta^*}{\eta^* + |z_1 - z_2|^2} \right)^{1/4} \right) \end{aligned} \quad (6.9)$$

holds with very high probability uniformly in $\eta_* \geq |z_1 - z_2|^4 \vee n^{-1/3}$. Note that in the last inequality in (6.9) we used that $\mathcal{E}(n, \eta_{1,0}, \eta_{2,0}) = (n\eta_{1,0}\eta_{2,0})^{-1} \lesssim [n\eta_*(\eta^* + |z_1 - z_2|^2)]^{-1}$ since $|\eta_{i,0}| \geq |z_1 - z_2|^2$. Along the flow z_1 and z_2 also move slightly, but it is easy to check that $|z_{1,t} - z_{i,t}| \sim |z_1 - z_2|$ for any $0 \leq t \leq T$ since $T \ll |z_1 - z_2|$ and the speed of $z_{i,t}$ is bounded. Therefore we ignored the difference between $|z_{1,t} - z_{i,t}|$ and $|z_1 - z_2|$ in the above estimates.

The induction step is now exactly as above using $\tilde{\eta}^{(k)} := [|z_1 - z_2|^{2(k+1)} \vee n^{-1+(2/3)^k}] \vee n^{-1+\epsilon}$ instead of $\eta^{(k)}$ at each induction step and using

$$\mathcal{E}(n, \eta_{1,0}, \eta_{2,0}) = \frac{n^{3k\xi}}{n\eta_{*,0}^{3/2}(\eta_0^*)^{1/2}} \left(\eta_{*,0}^{1/6} + n^{-1/10} + \frac{1}{\sqrt{n\eta_{*,0}}} + \left(\frac{\eta_0^*}{\eta_0^* + |z_1 - z_2|^2} \right)^{1/4} \right) \quad (6.10)$$

for $T = \tilde{\eta}^{(k)}$ and $|\eta_{i,0}| \geq \tilde{\eta}^{(k)}$ as an input for Part 2 of Proposition 5.3. The application of Proposition 6.2 is completely analogous to the case $\eta^* \geq |z_1 - z_2|^2$ in the first part of the proof. In particular, at the $k + 1$ -th step we get that for

$\eta_* \geq \tilde{\eta}^{(k+1)}$ it holds

$$\begin{aligned}
| \langle (G^{z_1}(i\eta_1)AG^{z_2}(i\eta_2) - M_{12}^A)B \rangle | &\lesssim \frac{n^{3k\xi}}{n\eta_*\eta^*} \left(\eta_*^{1/6} + n^{-1/10} + \left(\frac{\eta^*}{\eta^* + |z_1 - z_2|^2} \right)^{1/4} \right) \\
&\quad + \frac{n^{2\xi}}{n\sqrt{|\eta_1\eta_2|\eta_{*,T}(\eta^* + |z_1 - z_2|^2)}} + \frac{n^{3\xi}}{(n\eta_*)^{3/2}\sqrt{|\eta_1\eta_2|}} \\
&\quad + n^{2\xi}\mathcal{E}(n, \eta_{1,0}, \eta_{2,0}) \\
&\lesssim \frac{n^{3(k+1)\xi}}{n\eta_*^{3/2}(\eta^*)^{1/2}} \left(\eta_*^{1/6} + n^{-1/10} + \frac{1}{\sqrt{N}\eta_*} + \left(\frac{\eta^*}{\eta^* + |z_1 - z_2|^2} \right)^{1/4} \right), \tag{6.11}
\end{aligned}$$

where we used the expression of $\mathcal{E}(n, \eta_{1,0}, \eta_{2,0})$ from (6.10). In the last inequality of (6.11) we used the definition of $\tilde{\eta}^{(k)}$ and the fact that $|\eta_{i,t}|$ is decreasing in time to show that $\mathcal{E}(n, \eta_{1,0}, \eta_{2,0})$ is smaller than the last line of (6.11) (recall that $\eta_i = \eta_{i,T}$). Iterating this procedure $\sim 1/(2\omega_d) \vee |\log \epsilon|$ times we conclude the proof of this theorem in the case $\eta^* < |z_1 - z_2|^2$ as well. We remark that, also in this case, the accumulating factors of n^ξ is not a problem, since the exponent ξ can always be redefined before every iteration step and the number of iteration steps is finite. \square

We conclude this section with the proofs of Propositions 6.1–6.2.

Proof of Proposition 6.2. By (6.4), estimating $n^{-2/k-1/2} \leq n^{-1/10}$ for any $k \geq 5$ in the first term of (6.4), we get

$$|d \mathbf{E} |R_t|^p| \lesssim \left(\frac{n^\xi}{n\eta_1\eta_2} \right)^4 |R_t|^{p-4} + \sum_{k=5}^p \left(\frac{n^{-1/10+\xi}}{n\eta_1\eta_2} \right)^k + \left(1 + \frac{1}{\sqrt{n}\eta_*} \right) \mathbf{E} \left[\frac{|R_t|^{p-1}}{n\eta_1\eta_2} + |R_t|^{p-3} \left(\frac{1}{n\eta_1\eta_2} \right)^3 \right]. \tag{6.12}$$

We now estimate

$$\begin{aligned}
\left(\frac{n^\xi}{n\eta_1\eta_2} \right)^r |R_t|^{p-r} &\leq \left(\frac{\eta^* + |z_1 - z_2|^2}{\eta^*} \right) \left(\frac{n^\xi}{n\eta_*(\eta^*)^{1-1/r}(|z_1 - z_2|^2 + \eta^*)^{1/r}} \right)^r |R_t|^{p-r}, \\
\frac{1}{\sqrt{n}\eta_*} \left(\frac{1}{n\eta_1\eta_2} \right)^r |R_t|^{p-r} &\leq \frac{1}{\sqrt{n}\eta_*^{3/2}} \left(\frac{\eta_*^{1/2r}}{n\eta_1\eta_2} \right)^r |R_t|^{p-r}. \tag{6.13}
\end{aligned}$$

Then, combining (6.12) with (6.13), where we use the first bound for $r = 1, 3, 4$ and the second bound for $r = 1, 3$, and using that $\eta^* \leq 1$, $(\eta^* + |z_1 - z_2|^2)/\eta^* \geq 1$ we conclude

$$|d \mathbf{E} |R_t|^p| \lesssim \left(\left(\frac{\eta^* + |z_1 - z_2|^2}{\eta^*} \right)^4 + \frac{1}{\sqrt{n}\eta_*^{3/2}} \right) \left(|R_t|^p + \left(\frac{n^\xi \eta_*^{1/6} + n^{-1/10+\xi}}{n\eta_*\eta^*} \right)^p + \left(\frac{n^\xi}{n\eta_*(|z_1 - z_2|^2 + \eta^*)} \right)^p \right). \tag{6.14}$$

Finally, (6.5) readily follows from (6.14) by a simple Gronwall inequality. \square

Proof of Proposition 6.1. For notational simplicity we drop the absolute value. In fact the whole proof verbatim applies to $\prod_{k=1}^p R_t(\sigma_{k,1}\eta_1, \sigma_{k,2}\eta_2)$ for any $\sigma_{k,1}, \sigma_{k,2} = \pm 1$ and therefore the absolute value can be obtained by choosing half the σ 's to be +1 and the other half -1 with A, B replaced by B^*, A^* .

By Itô's formula we have

$$d \mathbf{E} R_t^p = \mathbf{E} \left[-\frac{1}{2} \sum_{\alpha} w_{\alpha}(t) \partial_{\alpha} R_t^p + \frac{1}{2} \sum_{\alpha, \beta} \kappa(\alpha, \beta) \partial_{\alpha} \partial_{\beta} R_t^p \right], \tag{6.15}$$

where $w_\alpha(t)$ are the entries of W_t for double-indices $\alpha = (a, b) \in [2n]^2$, and $\kappa(\alpha, \beta)$ denotes the joint cumulant of w_α, w_β . Performing a cumulant expansion¹⁰ in (6.15) we conclude that

$$d \mathbf{E} R_t^p = -\frac{1}{2} \sum_{k=3}^K \sum_{ab} \sum_{\alpha \in \{ab, ba\}^k} \frac{\kappa(\alpha)}{(k-1)!} \mathbf{E} \partial_\alpha R_t^p + \Omega_K \quad (6.16)$$

with an error term $\Omega_K = \mathcal{O}_\prec((n\eta_1\eta_2)^{-p})$ for $K = \text{const} \cdot p$, where we recall from (5.2) that \sum_{ab} is a shorthand notation for $\sum_{a \leq n} \sum_{b > n} + \sum_{a > n} \sum_{b \leq n}$. We first establish a priori bounds on derivatives of R_t which are sufficient for higher order cumulants.

Lemma 6.3. *For any $\alpha \in \{ab, ba\}^k$ and $k \geq 1$ we have*

$$|\partial_\alpha R_t| \prec \frac{1}{n\eta_1\eta_2}. \quad (6.17)$$

Proof. By the differentiation identity $\partial_{ab}G = -G\Delta^{ab}G$ for the resolvent (which follows from applying the resolvent identity to the difference quotient) we have that

$$\partial_{ab}R_t = -\frac{(G_1AG_2BG_1)_{ba} + (G_2BG_1AG_2)_{ba}}{n} \quad (6.18)$$

and therefore any derivative is a sum of monomials of types

$$(G_1AG_2BG_1) \prod G, \quad (G_2BG_1AG_2) \prod G, \quad \text{or} \quad (G_1AG_2)(G_2BG_1) \prod G,$$

where $\prod G$ stands for some product of entries of either G_1, G_2 . By estimating $|G_{ab}| \prec 1$ (see (3.6)) and Schwarz-inequalities of the form

$$\begin{aligned} |(G_1AG_2)_{ab}| &\leq \sqrt{(G_2^*A^*AG_2)_{bb}(G_1G_1^*)_{aa}} \leq \frac{\|A\| \sqrt{(\Im G_2)_{bb}(\Im G_1)_{aa}}}{\sqrt{\eta_1\eta_2}} \prec \frac{1}{\sqrt{\eta_1\eta_2}} \\ |(G_1AG_2BG_1)_{ab}| &\leq \sqrt{(G_1^*B^*G_2^*A^*AG_2BG_1)_{bb}(G_1G_1^*)_{aa}} \leq \frac{\|A\| \|B\| \sqrt{(\Im G_1)_{aa}(\Im G_1)_{bb}}}{\eta_1\eta_2} \prec \frac{1}{\eta_1\eta_2} \end{aligned} \quad (6.19)$$

the claim follows. Here we also used the operator-bound $0 \leq A^*A \leq \|A\|^2 I$ and the Ward identity $GG^* = \Im G/\eta$. \square

By distributing the derivatives (6.16) according to the Leibniz rule and using Lemma 6.3 it follows that we can estimate the k -th order terms using $|\kappa(\alpha)| \lesssim N^{-k/2}$ by

$$n^{2-k/2} \sum_{l=1}^{k \wedge p} \left(\frac{1}{n\eta_1\eta_2} \right)^l |R_t|^{p-l}. \quad (6.20)$$

We now consider the third order terms ($k = 3$) which symbolically (with $\partial = \partial_{ab} + \partial_{ba}$ and $R = R_t$ and ignored summations and constants) are given by

$$(\partial^3 R)R^{p-1} + (\partial^2 R)(\partial R)R^{p-2} + (\partial R)^3 R^{p-3}. \quad (6.21)$$

We begin with the last term in (6.21) which is given by

$$\begin{aligned} n^{-3/2} |(\partial R)^3| &\lesssim n^{-9/2} \sum_{ab} |(G_1AG_2BG_1)_{ba} + (G_2BG_1AG_2)_{ba}|^3 \\ &\prec \frac{1}{n^{9/2}\eta_1\eta_2} \sum_{ab} \left(|(G_1AG_2BG_1)_{ba}|^2 + |(G_2BG_1AG_2)_{ba}|^2 \right) \\ &= \frac{1}{n^{7/2}} \left(\frac{\langle \Im G_1AG_2B\Im G_1B^*G_2^*A^* \rangle}{\eta_1^3\eta_2} + \frac{\langle \Im G_2BG_1A\Im G_2A^*G_1^*B^* \rangle}{\eta_1\eta_2^3} \right) \\ &\prec \frac{1}{n^{7/2}} \frac{\eta_*}{\eta_*^2(\eta_1\eta_2)^3} = \left(\frac{1}{n\eta_1\eta_2} \right)^3 \frac{1}{\sqrt{n\eta_*}}, \end{aligned} \quad (6.22)$$

¹⁰Such an expansion was first used in the random matrix context in [31] and later revived in [29, 36]. Technically we use a truncated version of the expansion above, see e.g. [24, 29, 24]. The truncation error of the cumulant expansion after $K = \text{const} \cdot p$ terms can be estimated trivially by the single- G local law for resolvent entries, and by norm for entries of $GAG \cdots$ resolvent chains.

where in the second step we used the entrywise estimate from (6.19) for one of the factors, and in the third step performed the a, b summation for the other two. The power counting behind (6.22) is rather simple: the naive size of $(\partial R)^3$ would be given by $n^{-3/2} \times n^2 \times (n\eta_1\eta_2)^{-3}$ if all three factors were estimated by the a priori estimate in (6.17), where the $n^{-3/2}$ comes from the third order cumulant, n^2 from the summation, and $(n\eta_1\eta_2)^{-3}$ from the a priori estimate in (6.17). However, by summation off-diagonal resolvent chains, i.e. products of resolvents and deterministic matrices evaluated in the (a, b) or (b, a) -entry, can be estimated more efficiently. By two *Ward estimates* we can gain an additional factor of $[(n\eta_*)^{-1/2}]^2$ over the naive size, giving $(n\eta_1\eta_2)^{-3}n^{-1/2}\eta_*^{-1}$ for the final estimate. The very same power counting principle applies also to the first two terms in (6.21) whenever at least two off-diagonal resolvent chains appear (in which case all three resolvent chains are off-diagonal by parity) that allows us to sum up a, b into a full trace. Now we record the remaining terms explicitly:

$$\begin{aligned} n^{-3/2} |(\partial^3 R)| &\prec \frac{(\sqrt{n}\eta_*)^{-1}}{n\eta_1\eta_2} + \left| \sum_{ab} \frac{(GAGBG)_{ab}G_{aa}G_{bb} + (GAG)_{ab}(GBG)_{aa}G_{bb} + G_{ab}(GAG)_{aa}(GBG)_{bb}}{n^{5/2}} \right| \\ n^{-3/2} |(\partial^2 R)(\partial R)| &\prec \frac{(\sqrt{n}\eta_*)^{-1}}{(n\eta_1\eta_2)^2} + \left| \sum_{ab} \frac{(GAGBG)_{ab} \left((GAGBG)_{aa}G_{bb} + (GAG)_{aa}(GBG)_{bb} \right)}{n^{7/2}} \right| \end{aligned} \quad (6.23)$$

where we dropped the subscripts from G for notational brevity as they play no role in the sequel. For the terms in (6.23) with some G_{aa} or G_{bb} we further split $G_{aa} = (G - m)_{aa} + m$ so that using the entry-wise local law $|(G - m)_{aa}| \prec (n\eta_*)^{-1/2}$ from (3.6) gives the same estimate as the first term on the right hand side of (6.23)¹¹. For the m contribution we use a so called *isotropic resummation* trick: for example for the second line of (6.23) after splitting $G_{bb} = m + (G - m)_{bb}$ for the m -contribution the b -index only appears once in $(GAGBG)_{ab}$ and can be summed into the inner product $\langle e_a, GAGBG\mathbf{1}_a \rangle$ with the a -th unit vector e_a and the constant- $(0, 1)$ vector $\mathbf{1}_a$ defined as $\mathbf{1}_a := (0, \dots, 0, 1, \dots, 1)$ for $a > n$ and $\mathbf{1}_a := (1, \dots, 1, 0, \dots, 0)$ for $a \leq n$ (recall that the \sum_{ab} summation is restricted to either $a \leq n, b > n$ or $a > n, b \leq n$). Thus

$$\begin{aligned} &\left| \sum_{ab} \frac{(GAGBG)_{ab}(GAGBG)_{aa}}{n^{7/2}} \right| \\ &= \left| \sum_a \frac{\langle e_a, GAGBG\mathbf{1}_a \rangle (GAGBG)_{aa}}{n^{7/2}} \right| \\ &\prec \frac{\sqrt{\langle \mathbf{1}_n, G^*B^*G^*A^*G^*E_1GAGBG\mathbf{1}_n \rangle} + \sqrt{\langle \mathbf{1}_{2n}, G^*B^*G^*A^*G^*E_2GAGBG\mathbf{1}_{2n} \rangle}}{n^3\eta_1\eta_2} \\ &\leq \frac{\|A\|\|B\|\sqrt{\langle \mathbf{1}_n, \Im G\mathbf{1}_n \rangle} + \langle \mathbf{1}_{2n}, \Im G\mathbf{1}_{2n} \rangle}}{n^3\eta_1^2\eta_2^2\sqrt{\eta_1}} \prec \left(\frac{1}{n\eta_1\eta_2} \right)^2 \frac{1}{\sqrt{n\eta_*}} \end{aligned} \quad (6.24)$$

using Cauchy-Schwarz, the operator bound $B^*G^*A^*G^*E_1GAGB \lesssim (\|B\|^2\|A\|^2\eta^{-4})I$ and the *isotropic* local law $\langle \mathbf{1}_a, \Im G\mathbf{1}_a \rangle \prec \|\mathbf{1}_a\|^2 = n$. The same argument applies to the diagonal G_{aa}, G_{bb} terms in the first line of (6.23). Together with the following Lemma 6.4 for the remaining terms of (6.23) we thus conclude the proof of Proposition 6.1. \square

Lemma 6.4.

$$\left| n^{-5/2} \sum_{ab} \mathbf{E} G_{ab}(GAG)_{aa}(GBG)_{bb} \right| \prec \frac{1}{n\eta_1\eta_2} \left(1 + \frac{1}{\sqrt{n\eta_*}} \right), \quad (6.25)$$

$$\left| n^{-7/2} \sum_{ab} \mathbf{E} (GAGBG)_{ab}(GAG)_{aa}(GBG)_{bb} \right| \prec \left(\frac{1}{n\eta_1\eta_2} \right)^2 \left(1 + \frac{1}{\sqrt{n\eta_*}} \right) \quad (6.26)$$

¹¹For instance, for the $(GAGBG)(GAGBG)(G - m)$ term we obtain a bound of

$$n^{-7/2}\eta^{-2}(n\eta)^{-1/2} \sum_{ab} |(GAGBG)_{ab}| \leq n^{-5/2}\eta^{-5/2} \langle |GAGBG|^2 \rangle^{1/2} \prec n^{-5/2}\eta^{-5} = (n\eta^2)^{-2}(\sqrt{n\eta})^{-1},$$

ignoring the difference of η_1, η_2 for convenience.

Proof of Lemma 6.4. For (6.25) we write

$$G = M - M\underline{W}G + G\mathcal{O}_{\prec}\left(\frac{1}{n\eta_*}\right) := M - M(WG + \mathcal{S}[G]G) + G\mathcal{O}_{\prec}\left(\frac{1}{n\eta_*}\right), \quad (6.27)$$

c.f. [13, Eq.(5.2)], in terms of the “underline renormalization” from (A.4). When using (6.27) for G_{ab} in (6.25) the M - and $\mathcal{O}_{\prec}(\dots)$ -contributions are trivially of the claimed size. For the $\underline{W}G$ -part we compute with another cumulant expansion

$$\begin{aligned} \left| n^{-5/2} \sum_{ab} \mathbf{E}(\underline{W}G)_{ab} (GAG)_{aa} (GBG)_{bb} \right| &\lesssim n^{-7/2} \left| \sum_{abc} \mathbf{E} G_{cb} \partial_{ca} \left((GAG)_{aa} (GBG)_{bb} \right) \right| \\ &+ n^{-5/2} \sum_{k \geq 2} \left| \sum_{abc} \sum_{\beta \in \{ac, ca\}^k} \frac{\kappa(ac, \beta)}{k!} \mathbf{E} \partial_{\beta} \left(G_{cb} (GAG)_{aa} (GBG)_{bb} \right) \right|. \end{aligned} \quad (6.28)$$

For the last term the trivial estimate is sufficient: the ∂_{β} -derivative is bounded by $(\eta_1 \eta_2)^{-1}$ c.f. (6.19), while the prefactor, i.e. the cumulant and the summation, contribute $n^{-k/2}$. For the first term we exploit that the action of the ∂_{ca} derivative yields at least one additional off-diagonal resolvent chain, hence we gain an additional factor of $[(n\eta_*)^{-1/2}]^2$ over the naive size $n^{-1/2}(\eta_1 \eta_2)^{-1}$ to obtain a final bound of $n^{-1/2}(\eta_1 \eta_2)^{-1} (n\eta_*)^{-1}$, just as claimed. More explicitly, if for instance the derivative acts on the second G factor, then we estimate

$$\begin{aligned} n^{-7/2} \sum_{abc} |G_{cb} (GAG)_{ac} G_{aa} (GBG)_{bb}| &\leq n^{-7/2} \sum_{ab} |G_{aa}| |(GBG)_{bb}| \sqrt{\sum_c |G_{cb}|^2} \sqrt{\sum_c |(GAG)_{ac}|^2} \\ &= n^{-7/2} \sum_{ab} |G_{aa}| |(GBG)_{bb}| \sqrt{(G^*G)_{bb} (GAGG^*A^*G^*)_{aa}} \\ &\prec n^{-3/2} \eta^{-3}, \end{aligned}$$

just as claimed. We point out that in the last inequality we used (6.19) to estimate all the factors in the second to last line.

In order to prepare the more complicated estimate on (6.26) we now explain the general power counting principle behind the improvement in (6.28): the trivial estimate on the left hand side of (6.25) using one off-diagonal gain for G_{ab} is by a factor of $\sqrt{n\eta_*}$ larger than the right hand side. Performing the cumulant expansion for $G = M - M\underline{W}G + \dots$ is neutral for the Gaussian (i.e. second order cumulant) term (one additional summation is compensated by $\kappa(ac, ca) = 1/n$) but gains a factor of $n^{-1/2}$ already for the third order cumulants. This and the additional gain in the Gaussian term due to the second off-diagonal resolvent chain is sufficient for the claimed bound.

We now consider (6.26) where the non-Gaussian terms, as well as the Gaussian terms with two off-diagonal resolvent chains, of the cumulant expansion can be estimated just as for (6.25) using either the gain from the third cumulant or the additional Ward estimate. Thus we obtain

$$\begin{aligned} &\left| n^{-7/2} \sum_{ab} \mathbf{E}(\underline{W}GAGBG)_{ab} (GAG)_{aa} (GAG)_{bb} \right| \\ &\lesssim \left(\frac{1}{n\eta_1 \eta_2} \right)^2 \frac{1}{\sqrt{n\eta_*}} + \left| n^{-9/2} \sum_{abc} \mathbf{E}[(GAGBG)_{cc} G_{ab} + (GAG)_{cc} (GBG)_{ab}] (GAG)_{aa} (GBG)_{bb} \right|. \end{aligned} \quad (6.29)$$

We now continue with another cumulant expansion (6.27) for $G_{ab} = M_{ab} - (\underline{MWG})_{ab} + \dots$ and $(GBG)_{ab} = (MBG)_{ab} - (\underline{MWGBG})_{ab} + \dots$ and obtain

$$\begin{aligned} & \left| n^{-9/2} \sum_{abc} \mathbf{E}(GAGBG)_{cc} (\underline{MWG})_{ab} (GAG)_{aa} (GAG)_{bb} \right| < \left(\frac{1}{n\eta_1\eta_2} \right)^2 \left(1 + \frac{1}{\sqrt{n\eta_*}} \right) \\ & \left| n^{-9/2} \sum_{abc} \mathbf{E}(GAG)_{cc} (\underline{MWGBG})_{ab} (GAG)_{aa} (GBG)_{bb} \right| \\ & < \left(\frac{1}{n\eta_1\eta_2} \right)^2 \left(1 + \frac{1}{\sqrt{n\eta_*}} \right) + \left| n^{-11/2} \sum_{abcd} \mathbf{E}(GAG)_{cc} (GBG)_{dd} (MG)_{ab} (GAG)_{aa} (GBG)_{bb} \right| < \left(\frac{1}{n\eta_1\eta_2} \right)^2 \left(1 + \frac{1}{\sqrt{n\eta_*}} \right), \end{aligned} \quad (6.30)$$

where for the first inequality we used that for the Gaussian term the ∂_{da} -derivative creates one additional off-diagonal resolvent chain. For the second inequality we kept the unique term in which the ∂_{da} derivative does not create an additional off-diagonal resolvent chain and estimated the remaining terms as before. Finally, for the last inequality we performed another cumulant expansion in $G_{ab} = M_{ab} - (\underline{MWG})_{ab} + \dots = M_{ab} - n^{-1} \sum_e G_{eb} \partial_{ea}[\dots] + \dots$ and used that the ∂_{ea} -derivative creates a second off-diagonal resolvent chain which is sufficient to achieve the claimed bound. Finally the M - and $\mathcal{O}_<(\cdot)$ - contributions of (6.27) towards (6.30) can be trivially estimated. \square

APPENDIX A. ADDITIONAL TECHNICAL RESULTS

Here we prove several technical inputs which are used in Section 5.

A.1. Proof of Proposition 3.4. The proof is essentially identical to the proof of [13, Proposition 3.3] upon replacing [13, Theorem 5.2] by the improved Theorem 3.3. For the sake of brevity we here only give a sketch highlighting the differences. The analogue of [13, Eq. (6.9)] is

$$\langle G - M \rangle = \langle \mathcal{D}^{-1}[I]M[-\underline{WG} + \mathcal{S}G - M] \rangle, \quad \mathcal{D}[R] := 1 - \mathcal{S}[MRM] \quad (A.1)$$

and by an explicit computation it follows that

$$\mathcal{D}^{-1}[I] = \frac{I}{1 - m^2 - |z|^2 u^2}, \quad \|\mathcal{D}^{-1}[I]\| \lesssim \frac{1}{\tau} \lesssim 1 \quad (A.2)$$

and therefore

$$\langle G - M \rangle = -\langle A\underline{WG} \rangle + \mathcal{O}_<\left(\frac{1}{(n\eta)^2}\right), \quad A := \frac{M}{1 - m^2 - |z|^2 u^2}. \quad (A.3)$$

Here

$$\underline{Wf}(W) := Wf(W) - \widetilde{\mathbf{E}}\widetilde{W}(\partial_{\widetilde{W}}f)(W) \quad (A.4)$$

for any given f , where \widetilde{W} is an independent copy of W , and $\partial_{\widetilde{W}}$ denotes the directional derivative in the direction \widetilde{W} . In particular, we have

$$\underline{WG} = WG + \mathcal{S}[G]G.$$

The accuracy of the expectation computation from [13, Lemma 6.2] was already sufficient also on optimal mesoscopic scales and we recall from (3.17) that

$$\langle G - \mathbf{E}G \rangle = \langle G - M - \mathcal{E} \rangle + \mathcal{O}_<\left(\frac{1}{n^{3/2}(1+\eta)} + \frac{1}{(n\eta)^2}\right), \quad (A.5)$$

where

$$\mathcal{E} := -\frac{i\kappa_4}{4n}\partial_\eta(m^4) = \frac{\kappa_4}{n}m^3\langle MA \rangle = \frac{\kappa_4}{n} \frac{m^3\langle M^2 \rangle}{1 - m^2 - |z|^2 u^2} \quad (A.6)$$

using [13, Eq. (6.10)].

Therefore, for the higher moments it suffices to compute

$$\mathbf{E} \prod_i \langle G_i - \mathbf{E}G_i \rangle = \prod_i \langle -A_i \underline{WG}_i - \mathcal{E}_i \rangle + \mathcal{O}_<\left(\frac{\psi}{\sqrt{n\eta_*}}\right), \quad \psi := \prod_i \frac{1}{n|\eta_i|} \quad (A.7)$$

due to

$$|\langle A_i \underline{W} G_i \rangle| \prec \frac{1}{n\eta_i} \quad (\text{A.8})$$

by [13, Eq. (6.14)]. As in [13, Eq. (6.17)] we then perform a cumulant expansion to obtain

$$\begin{aligned} & \mathbf{E} \prod_i \langle -A_i \underline{W} G_i - \mathcal{E}_i \rangle \\ &= -\langle \mathcal{E}_i \rangle \mathbf{E} \prod_{i \neq 1} \langle -A_i \underline{W} G_i - \mathcal{E}_i \rangle \\ &+ \sum_{i \neq 1} \mathbf{E} \tilde{\mathbf{E}} \langle -A_1 \tilde{W} G_1 \rangle \langle -A_i \tilde{W} G_i + A_i \underline{W} G_i \tilde{W} G_i \rangle \prod_{j \neq 1, i} \langle -A_j \underline{W} G_j - \mathcal{E}_j \rangle \\ &+ \sum_{k \geq 2} \sum_{ab} \sum_{\alpha \in \{ab, ba\}^k} \frac{\kappa(ba, \alpha)}{k!} \mathbf{E} \partial_\alpha \left[\langle -A_1 \Delta^{ba} G_1 \rangle \prod_{i \neq 1} \langle -A_i \underline{W} G_i - \mathcal{E}_i \rangle \right] \\ &= \sum_{i \neq 1} \mathbf{E} \left[\tilde{\mathbf{E}} \langle -A_1 \tilde{W} G_1 \rangle \langle -A_i \tilde{W} G_i + A_i \underline{W} G_i \tilde{W} G_i \rangle + \frac{\kappa_4 U_1 U_i}{2n^2} \right] \prod_{j \neq 1, i} \langle -A_j \underline{W} G_j - \mathcal{E}_j \rangle + \mathcal{O}_\prec \left(\frac{\psi}{\sqrt{n\eta_*}} \right). \end{aligned} \quad (\text{A.9})$$

with U_i as in (3.16), and where we used [13, Eqs. (6.26), (6.29)] for the last equality. By combining [13, Eq. (6.2)] and the display below [13, Eq. (6.21)] we have

$$\begin{aligned} & \mathbf{E} \tilde{\mathbf{E}} \langle -A_1 \tilde{W} G_1 \rangle \langle -A_i \tilde{W} G_i + A_i \underline{W} G_i \tilde{W} G_i \rangle \prod_{j \neq 1, i} \langle -A_j \underline{W} G_j - \mathcal{E}_j \rangle \\ &= \frac{\langle G_1 A_1 E G_i A_i E' + G_1 \mathcal{S}[G_1 A_1 E G_i A_i] G_i E' \rangle}{2n^2} \mathbf{E} \prod_{j \neq 1, i} \langle -A_j \underline{W} G_j - \mathcal{E}_j \rangle + \mathcal{O} \left(\frac{n^\epsilon \psi}{n\eta_*} \right), \end{aligned} \quad (\text{A.10})$$

where it is understood that (E, E') is summed over (E_1, E_2) and (E_2, E_1) . From Theorem 3.3 and the computations around [13, Eq. (6.23)] we obtain

$$\begin{aligned} \frac{\langle G_1 A_1 E G_i A_i E' + G_1 \mathcal{S}[G_1 A_1 E G_i A_i] G_i E' \rangle}{2n^2} &= \frac{\langle M_{A_1 E}^{z_1, z_i} + M_{E'}^{z_i, z_1} \mathcal{S}[M_{A_1 E}^{z_1, z_i}] \rangle}{2n^2} + \mathcal{O}_\prec \left(\frac{1}{n^2 \eta_1 \eta_i} \frac{1}{n\eta_*} \right) \\ &= \frac{V_{1, i}}{2n^2} + \mathcal{O}_\prec \left(\frac{1}{n^2 \eta_1 \eta_i} \frac{1}{n\eta_*} \right) \end{aligned} \quad (\text{A.11})$$

with $V_{1, i}$ as in (3.16). Inserting (A.10) and (A.11) into (A.9) we conclude the proof of Proposition 3.4 by induction.

A.2. Asymptotic independence. In this section we present the proof of Theorem 3.1 and of Proposition 3.5.

Proof of Theorem 3.1. Using the spectral symmetry of H^z , for any $z \in \mathbf{C}$ we write G^z in spectral decomposition as

$$G^z(i\eta) = \sum_{j > 0} \frac{2}{(\lambda_j^z)^2 + \eta^2} \begin{pmatrix} i\eta \mathbf{u}_j^z (\mathbf{u}_j^z)^* & \lambda_j^z \mathbf{u}_j^z (\mathbf{v}_j^z)^* \\ \lambda_j^z \mathbf{v}_j^z (\mathbf{u}_j^z)^* & i\eta \mathbf{v}_j^z (\mathbf{v}_j^z)^* \end{pmatrix}.$$

Let $\eta = n^{-1+\epsilon}$, with $\epsilon \leq \omega_p/10$, then by rigidity of the eigenvalues (see e.g. [13, Eq. (7.4)]), for any $1 \leq i_0, j_0 \leq n_B^\omega$ such that $\lambda_{i_0}^{z_1}, \lambda_{j_0}^{z_2} \lesssim \eta$, with $l = 1, 2$, and any z_1, z_2 such that $n^{-1/2+\omega_p} \leq |z_1 - z_2|^2 \leq n^{-2\omega_d}$, it follows that

$$\begin{aligned} & |\langle \mathbf{u}_{i_0}^{z_1}, \mathbf{u}_{j_0}^{z_2} \rangle|^2 + |\langle \mathbf{v}_{i_0}^{z_1}, \mathbf{v}_{j_0}^{z_2} \rangle|^2 \\ & \lesssim \sum_{i, j=1}^n \frac{4\eta^4}{((\lambda_i^{z_1})^2 + \eta^2)((\lambda_j^{z_2})^2 + \eta^2)} \left(|\langle \mathbf{u}_i^{z_1}, \mathbf{u}_j^{z_2} \rangle|^2 + |\langle \mathbf{v}_i^{z_1}, \mathbf{v}_j^{z_2} \rangle|^2 \right) \\ & = \eta^2 \text{Tr}(\Im G^{z_1}) (\Im G^{z_2}) \lesssim \frac{n\eta^2}{|z_1 - z_2|^2} + n^\xi \left(\eta^{1/6} + n^{-1/10} + \frac{1}{\sqrt{n\eta}} + \left(\frac{\eta}{|z_1 - z_2|^2} \right)^{1/4} \right) \\ & \lesssim n^{-2\delta} + n^{(\delta+2\xi-\omega_p)/2} + n^{-(\delta+\omega_p-4\xi)/4}. \end{aligned} \quad (\text{A.12})$$

The first inequality in the third line of (A.12) is from Theorem 3.3. In the last inequality we chose $\eta = n^{-1/2-\delta}|z_1 - z_2|$, for any $\delta \leq \omega_p/10$. This concludes the proof by choosing ω_B small enough and $\omega_E := (2\delta) \wedge [(\omega_p - \delta - 2\xi)/2] \wedge [(\delta + \omega_p - 4\xi)/4]$ which is still positive by our choice of δ and ξ . \square

Proof of Proposition 3.5. The proof of this proposition is completely analogous to [13, Proof of Proposition 3.5 in Section 7], the only difference is that [13, Lemma 7.9] has to be replaced by Theorem 3.1. The key change is that in [13, Proposition 3.5] we required $|z_1 - z_2|$ to basically be order one, whilst now we consider the entire mesoscopic regime $|z_1 - z_2| \gg n^{-1/2}$. In particular, Theorem 3.1 ensures that [13, Assumption 7.1] is satisfied even in the mesoscopic regime $|z_1 - z_2| \geq n^{-1/2+\omega_p}$ and so that [13, Proposition 7.14] holds in this case as well. This proposition and a simple standard GFT were the only input to prove [13, Proposition 3.5]. \square

A.3. Proof of Lemma 5.4. Without loss of generality we assume that $\|(\mathcal{B}_{12}^{-1})^*[B^*]\| \lesssim 1$. The case $\|(\mathcal{B}_{21}^{-1})^*[A^*]\| \lesssim 1$ is completely analogous using cyclicity of the trace.

We start writing down the equation for G_1AG_2 . First of all notice that

$$G = M - M\underline{W}G + M\mathcal{S}[G - M]G, \quad (\text{A.13})$$

where $W := H + Z$, with

$$Z := \begin{pmatrix} 0 & z \\ \bar{z} & 0 \end{pmatrix} = zF^* + \bar{z}F. \quad (\text{A.14})$$

Here we also recall that $\underline{W}G = WG + \langle G \rangle G$.

Next, using (A.13) for G_1 and writing $G_2 = M_2 + (G_2 - M_2)$, we find

$$\begin{aligned} G_1AG_2 &= M_1AM_2 + M_1A(G_2 - M_2) - M_1\underline{W}G_1AG_2 + M_1\mathcal{S}[G_1 - M_1]G_1AG_2 \\ &\quad + M_1\mathcal{S}[G_1AG_2]G_2 \\ &= M_1AM_2 + M_1A(G_2 - M_2) - M_1\underline{W}G_1AG_2 + M_1\mathcal{S}[G_1 - M_1]G_1AG_2 \\ &\quad + M_1\mathcal{S}[G_1AG_2]M_2 + M_1\mathcal{S}[G_1AG_2](G_2 - M_2), \end{aligned} \quad (\text{A.15})$$

where

$$\underline{W}G_1AG_2 := \underline{W}G_1AG_2 + \mathcal{S}[G_1AG_2]G_2.$$

Then, by (A.15) it follows that

$$\begin{aligned} \langle G_1AG_2B - M_{12}^AB \rangle &= \langle M_1A(G_2 - M_2)((\mathcal{B}_{12}^{-1})^*[B^*])^* \rangle - \langle M_1\underline{W}G_1AG_2((\mathcal{B}_{12}^{-1})^*[B^*])^* \rangle \\ &\quad + \langle M_1\mathcal{S}[G_1 - M_1]G_1AG_2((\mathcal{B}_{12}^{-1})^*[B^*])^* \rangle + \langle M_1\mathcal{S}[G_1AG_2](G_2 - M_2)((\mathcal{B}_{12}^{-1})^*[B^*])^* \rangle. \end{aligned} \quad (\text{A.16})$$

Finally, the single resolvent local law $|\langle G_i - M_i \rangle| \prec (n\eta_i)^{-1}$ from (3.6) and the bound

$$|\langle M_1\underline{W}G_1AG_2((\mathcal{B}_{12}^{-1})^*[B^*])^* \rangle| \prec \frac{1}{n\eta_*\sqrt{\eta_1\eta_2}}$$

from [13, Eq. (5.10c)] we conclude the proof of this lemma.

A.4. Proof of Lemma 5.5. We present the proof in the most general setting for convenience. Consider the matrix Dyson equation (MDE) $M = M(\Lambda)$ solving

$$-M^{-1} = \mathcal{S}[M] + \Lambda, \quad \text{sgn } \Im M = \text{sgn } \Im \Lambda \quad (\text{A.17})$$

for some generalised spectral parameter Λ with $\Im \Lambda$ either positive or negative definite in order for $\text{sgn } \Im \Lambda$ to be well defined.

Lemma A.1. *If $\Lambda = \Lambda(t) \in \mathbf{C}_{\pm}^{N \times N}$ (with \pm independent of t) solves the ODE*

$$\dot{\Lambda} := \frac{d\Lambda}{dt} = -\frac{\Lambda}{2} - \mathcal{S}[M], \quad (\text{A.18})$$

then

$$\dot{M} = \frac{M}{2}. \quad (\text{A.19})$$

Proof. By inverting and differentiating (A.17) we have

$$M^{-1}\dot{M}M^{-1} = \mathcal{S}[\dot{M}] + \dot{\Lambda} = \mathcal{S}[\dot{M}] - \mathcal{S}[M] - \frac{\Lambda}{2} = \mathcal{S}[\dot{M}] - \frac{\mathcal{S}[M]}{2} + \frac{M^{-1}}{2} \quad (\text{A.20})$$

and therefore

$$\mathcal{B}[\dot{M}] = \mathcal{B}\left[\frac{\dot{M}}{2}\right], \quad \mathcal{B} := 1 - M\mathcal{S}[\cdot]M \quad (\text{A.21})$$

and the claim follows upon inverting \mathcal{B} . \square

Lemma A.2. Let B be arbitrary, let $M_i = M(\Lambda_i)$ be the solution to (A.17) for $i = 1, 2$, and let

$$M_{12}^B = M^B(\Lambda_1, \Lambda_2) = \mathcal{B}_{12}^{-1}[M_1 B M_2], \quad \mathcal{B}_{12} := 1 - M_1 \mathcal{S}[\cdot] M_2. \quad (\text{A.22})$$

If $\Lambda_i = \Lambda_i(t)$ satisfy the ODE

$$\dot{\Lambda}_i = -\frac{\Lambda_i}{2} - \mathcal{S}[M_i], \quad (\text{A.23})$$

then

$$\dot{M}_{12}^B = \mathcal{B}_{12}^{-1}[M_{12}^B]. \quad (\text{A.24})$$

Proof. We invert \mathcal{B}_{12} in (A.22) and differentiate to obtain

$$\begin{aligned} \dot{M}_{12}^B - M_1 \mathcal{S}[\dot{M}_{12}^B] M_2 &= \dot{M}_1 (B + \mathcal{S}[M_{12}]) M_2 + M_1 (B + \mathcal{S}[M_{12}]) \dot{M}_2 \\ &= \dot{M}_1 M_1^{-1} M_{12} + M_{12} M_2^{-1} \dot{M}_2 = M_{12} \end{aligned} \quad (\text{A.25})$$

where we used (A.22) in the second step and Lemma A.1 in the final step. \square

Lemma A.2 implies for arbitrary B_1, B_2 that

$$\begin{aligned} \langle \dot{M}_{12}^{B_1} B_2 \rangle &= \langle B_2 \mathcal{B}_{12}^{-1}[M_{12}^{B_1}] \rangle = \langle (1 - \mathcal{S}[M_2 \cdot M_1])^{-1} [B_2] M_{12}^{B_1} \rangle \\ &= \langle M_2^{-1} M_{21}^{B_2} M_1^{-1} M_{12}^{B_1} \rangle = \langle B_2 M_{12}^{B_1} \rangle + \langle \mathcal{S}[M_{21}^{B_2}] M_{12}^{B_1} \rangle \end{aligned} \quad (\text{A.26})$$

using $M_2^{-1} M_{21}^{B_2} M_1^{-1} = \mathcal{S}[M_{21}^{B_2}] + B$ in the last step. This concludes the proof of Lemma 5.5.

A.5. Proof of Lemma 5.6. Adding and subtracting the deterministic approximations of $G_1 L'_- G_2$ and $G_2 L_- G_1$, and using the definition of \mathcal{S} in (3.8), we obtain

$$\begin{aligned} \langle \mathcal{S}[G_1 L'_- G_2] G_2 L_- G_1 \rangle &= \langle \mathcal{S}[M_{12}^{L'_-}] M_{21}^{L'_-} \rangle + \langle G_1 L'_- G_2 - M_{12}^{L'_-} \rangle \langle M_{21}^{L'_-} \rangle + \langle G_1 L'_- G_2 \rangle \langle G_2 L_- G_1 - M_{21}^{L'_-} \rangle \\ &\quad - \langle (G_1 L'_- G_2 - M_{12}^{L'_-}) E_- \rangle \langle M_{21}^{L'_-} E_- \rangle - \langle G_1 L'_- G_2 E_- \rangle \langle (G_2 L_- G_1 - M_{21}^{L'_-}) E_- \rangle. \end{aligned} \quad (\text{A.27})$$

In Appendix B we will derive various elementary facts about the eigenvectors of the stability operator. In particular, in (B.6) we will prove that $\eta_{1,t} \eta_{2,t} < 0$ we have

$$I = (1 + \mathcal{O}(|z_{1,t} - z_{2,t}|)) L_- + \mathcal{O}(|z_{1,t} - z_{2,t}|) L_+, \quad E_- = (1 + \mathcal{O}(|z_{1,t} - z_{2,t}|)) L_+ + \mathcal{O}(|z_{1,t} - z_{2,t}|) L_-, \quad (\text{A.28})$$

and similar relations hold with L_{\pm} replaced with L'_{\pm} . Next, using (A.28), we write

$$\langle (G_1 L'_- G_2 - M_{12}^{L'_-}) E_- \rangle = (1 + \mathcal{O}(|z_{1,t} - z_{2,t}|)) \langle (G_1 L'_- G_2 - M_{12}^{L'_-}) L_+ \rangle + \mathcal{O}(|z_{1,t} - z_{2,t}|) \langle (G_1 L'_- G_2 - M_{12}^{L'_-}) L_- \rangle.$$

We can thus estimate

$$\langle (G_1 L'_- G_2 - M_{12}^{L'_-}) E_- \rangle = \mathcal{O} \left(\frac{n^{\xi}}{n \eta_{*,t} \sqrt{|\eta_{1,t} \eta_{2,t}|}} + |z_1 - z_2| |Y_t| \right), \quad (\text{A.29})$$

where we used Lemma 5.4 for $A = L_+$ and $B = L'_-$ to estimate the terms containing L_+ . Notice that $\|(\mathcal{B}_{12}^*)^{-1} [L_+^*]\| = \|L_+\| / |\beta_+| \lesssim \|L_+\| \sim 1$, where we used that $|\beta_+| \sim 1$ by (B.3); this ensures the applicability Lemma 5.4 for $A = L_+$ and $B = L'_-$.

Using that

$$|\langle M_{21}^{L'_-} E_- \rangle| \lesssim |\langle M_{21}^{L'_-} L_+ \rangle| + |z_{1,t} - z_{2,t}| |\langle M_{21}^{L'_-} L_- \rangle| \lesssim 1 + \frac{|z_{1,t} - z_{2,t}|}{|z_{1,t} - z_{2,t}|^2 + \eta_{*,t}},$$

together with (A.29) and similar bounds for $\langle G_1 L'_- G_2 E_- \rangle = \langle M_{12}^{L'_-} E_- \rangle + \langle (G_1 L'_- G_2 - M_{12}^{L'_-}) E_- \rangle$, we obtain

$$\begin{aligned} \langle \mathcal{S}[G_1 L'_- G_2] G_2 L_- G_1 \rangle &= \langle \mathcal{S}[M_{12}^{L'_-}] M_{21}^{L'_-} \rangle + \langle G_1 L'_- G_2 - M_{12}^{L'_-} \rangle \langle M_{21}^{L'_-} \rangle + \langle G_1 L'_- G_2 \rangle \langle G_2 L_- G_1 - M_{21}^{L'_-} \rangle \\ &\quad + \mathcal{O} \left(\frac{n^\xi}{n \eta_{*,t}^2 \sqrt{|\eta_{1,t} \eta_{2,t}|}} + |Y_t| + |z_{1,t} - z_{2,t}|^2 |Y_t|^2 \right), \end{aligned} \quad (\text{A.30})$$

where we used that $\epsilon > \xi/10$ (recall $\eta_{*,t} \geq n^{-1+\epsilon}$).

Finally, writing $\langle G_1 L'_- G_2 \rangle = \langle M_{12}^{L'_-} \rangle + \langle (G_1 L'_- G_2 - M_{12}^{L'_-}) \rangle$, using

$$\begin{aligned} \langle M_{21}^{L'_-} \rangle &= \langle M_{12}^I \rangle (1 + \mathcal{O}(|z_{1,t} - z_{2,t}|)), \\ \langle G_1 L'_- G_2 - M_{12}^{L'_-} \rangle &= (1 + \mathcal{O}(|z_{1,t} - z_{2,t}|)) Y_t + \mathcal{O}(|z_{1,t} - z_{2,t}|) \langle (G_1 L'_- G_2 - M_{12}^{L'_-}) L_+ \rangle, \end{aligned} \quad (\text{A.31})$$

by (A.28), and a similar approximation for the last term in the first line of (A.30), we conclude (5.27).

A.6. Proof of Lemma 5.7. Define

$$\Xi_t = \Xi(\eta_{1,t}, \eta_{2,t}, z_{1,t}, z_{2,t}) := |\eta_{1,t}| + |\eta_{2,t}| + |z_{1,t} - z_{2,t}|^2.$$

Then, we estimate

$$\begin{aligned} \langle M_{12}^I \rangle &= \frac{m_{1,t} m_{2,t} + 1 - \Re[z_{1,t} \bar{z}_{2,t}] u_{1,t} u_{2,t}}{1 + |z_{1,t} z_{2,t}|^2 u_{1,t}^2 u_{2,t}^2 - m_{1,t}^2 m_{2,t}^2 - 2u_{1,t} u_{2,t} \Re[z_{1,t} \bar{z}_{2,t}]} - 1 \\ &= \frac{\sqrt{(1 - |z_{1,t}|^2)(1 - |z_{2,t}|^2)} + 1 - \Re[z_{1,t} \bar{z}_{2,t}]}{|z_{1,t} - z_{2,t}|^2 + |\eta_{1,t}| \sqrt{1 - |z_{1,t}|^2} + |\eta_{2,t}| \sqrt{1 - |z_{2,t}|^2}} (1 + \mathcal{O}(\Xi_t)) \\ &= \frac{\sqrt{(1 - |z_{1,0}|^2)(1 - |z_{2,0}|^2)} + 1 - \Re[z_{1,0} \bar{z}_{2,0}]}{|z_{1,0} - z_{2,0}|^2 + |\eta_{1,0}| \sqrt{1 - |z_{1,0}|^2} + |\eta_{2,0}| \sqrt{1 - |z_{2,0}|^2} - t[2 - |z_{1,0}|^2 - |z_{2,0}|^2]} (1 + \mathcal{O}(\Xi_0 + t)) \\ &\leq \frac{2 - |z_{1,0}|^2 - |z_{2,0}|^2}{|z_{1,0} - z_{2,0}|^2 + |\eta_{1,0}| \sqrt{1 - |z_{1,0}|^2} + |\eta_{2,0}| \sqrt{1 - |z_{2,0}|^2} - t[2 - |z_{1,0}|^2 - |z_{2,0}|^2]} (1 + \mathcal{O}(\Xi_0 + t)), \end{aligned} \quad (\text{A.32})$$

We point out that here we also used that

$$|\eta_{i,t}| = |\eta_{i,0}| - t \sqrt{1 - |z_{i,0}|^2} + \mathcal{O}(t |\eta_{i,t}|), \quad z_{i,t} = e^{-t/2} z_{i,0} = z_{i,0} (1 + \mathcal{O}(t)),$$

and that

$$1 - \Re[z_{1,0} \bar{z}_{2,0}] = 1 - \frac{|z_{1,0}|^2 + |z_{2,0}|^2}{2} + |z_{1,0} - z_{2,0}|^2.$$

Additionally, to go from the first to the second line of (5.30) we also used (3.7).

APPENDIX B. EIGENDECOMPOSITION OF THE STABILITY OPERATOR

The stability operator $\mathcal{B}_{12} := 1 - M_1 \mathcal{S}[\cdot] M_2$, with $M_i := M^{z_i}(w_i)$, acts on the $4n^2$ dimensional space of $(2n) \times (2n)$ block matrices. From the action of \mathcal{S} and the fact that M_i are block constant, it immediately follows that both \mathcal{B}_{12} and \mathcal{B}_{12}^* leave the $4n^2 - 4$ dimensional subspace of block traceless matrices invariant and they act trivially as the identity on it. Now we describe its spectral data on the 4 dimensional space of block constant matrices, which are a constant multiple of the $n \times n$ identity in all four blocks. We use the notation $u_i := u^{z_i}(i\eta_i)$, $m_i := m^{z_i}(i\eta_i)$, with u^z, m^z being defined in (3.4) and (3.5), respectively. There are eigenvalues $(1, 1, \beta_+, \beta_-)$ with¹²

$$\beta_\pm := 1 \pm \sqrt{s} - u_1 u_2 \Re z_1 \bar{z}_2, \quad s := m_1^2 m_2^2 - u_1^2 u_2^2 (\Im z_1 \bar{z}_2)^2, \quad (\text{B.1})$$

and right eigenvectors F, F^*, R_+, R_- in the sense that $\mathcal{B}_{12}[R_\pm] = \beta_\pm R_\pm$ and $\mathcal{B}_{12}[F^{(*)}] = F^{(*)}$, where F has been defined in (3.9) and

$$R_\pm = \begin{pmatrix} -u_1 u_2 \Re z_1 \bar{z}_2 \pm \sqrt{s} & z_1 u_1 m_2 + \frac{z_2 u_2 m_1^2 m_2}{i u_1 u_2 \Im z_1 \bar{z}_2 \mp \sqrt{s}} \\ \bar{z}_2 u_2 m_1 + \frac{\bar{z}_1 u_1 m_2^2 m_1}{i u_1 u_2 \Im z_1 \bar{z}_2 \mp \sqrt{s}} & \frac{m_1 m_2}{i u_1 u_2 \Im z_1 \bar{z}_2 \mp \sqrt{s}} (-u_1 u_2 \Re z_1 \bar{z}_2 \pm \sqrt{s}) \end{pmatrix}. \quad (\text{B.2})$$

¹²The complex square root $\sqrt{\cdot}$ in (B.1) and (B.2) is defined using the standard branch cut $\mathbf{C} \setminus (-\infty, 0)$.

We point out that the quantities s, β_{\pm}, R_{\pm} , and L_{\pm} below naturally depend on z_1, z_2, η_1, η_2 ; we omitted this from the notation for simplicity. Furthermore, we remark that the explicit formulas for the eigenvalues in (B.1), for the right eigenvectors in (B.2), and for the left eigenvectors in (B.4) below hold for any $z_1, z_2 \in \mathbf{C}$ and for any spectral parameters $w_1, w_2 \in \mathbf{C} \setminus \mathbf{R}$; however we present the following estimates only on the imaginary axis, i.e. for $w_i = i\eta_i$, and for $1 - |z_i|^2 \sim 1, |z_1 - z_2| \ll 1$, since this is the only regime needed in the proof of Theorem 2.1. Note that by an explicit computation for $z_1 = z_2$ and a simple Taylor expansion it follows that $\|R_{\pm}\| \sim 1$. On the eigenvalues β_{\pm} we also have the following asymptotics:

$$\beta_- \sim |z_1 - z_2|^2 + \eta_1 + \eta_2, \quad |\beta_+| \sim 1. \quad (\text{B.3})$$

The fact that $|\beta_+| \sim 1$ follows trivially by its explicit expression in (B.1), the lower bound for $|\beta_-|$ follows by [16, Lemma 6.1], whilst the upper bound follows from the fact that for $\eta_1 = \eta_2 = 0$ we have

$$\begin{aligned} \beta_- &= 1 - \Re[z_1 \bar{z}_2] - \sqrt{1 - |z_1|^2 - |z_2|^2 + \Re[z_1 \bar{z}_2]^2} \\ &= 1 - \frac{|z_1|^2 + |z_2|^2}{2} + \frac{|z_1 - z_2|^2}{2} - \sqrt{\left(1 - \frac{|z_1|^2 + |z_2|^2}{2}\right)^2 - |z_1 - z_2|^2 \left(|z_1|^2 + |z_2|^2 + \frac{|z_1 - z_2|^2}{4}\right)} \\ &\lesssim |z_1 - z_2|^2 \end{aligned}$$

and a simple Taylor expansion in the η_i variables. We point out that here we used $2\Re[z_1 \bar{z}_2] = |z_1|^2 + |z_2|^2 - |z_1 - z_2|^2$ and that for $\eta_i = 0$ it holds $m_i^2 = |z_i|^2 - 1, u_i = 1$.

Since \mathcal{B}_{12} is not self-adjoint, it has a separate set of left eigenvectors defined by

$$\mathcal{B}_{12}^*[L_{\pm}^*] = \overline{\beta_{\pm}} L_{\pm}^*, \quad \mathcal{B}_{12}^*[L_{(*)}^*] = L_{(*)}^*.$$

The left eigenvectors corresponding to $\overline{\beta_{\pm}}$ are given by

$$L_{\pm} = \frac{1}{m_1 m_2} \begin{pmatrix} i u_1 u_2 \Im z_1 \bar{z}_2 \mp \sqrt{s} & 0 \\ 0 & m_1 m_2 \end{pmatrix}. \quad (\text{B.4})$$

We do not give the explicit form of the eigenvectors $L_{(*)}^*$ since they are not used for our analysis. With the normalizations above we have $\|L_{\pm}\| \sim 1$, and $\langle L_- R_- \rangle \sim 1$; however $\langle L_+ R_+ \rangle$ can be small when $|z_1|, |z_2| \approx 2^{-1/2}$, i.e. when β_+ resonates with the eigenvalue 1. As before, for these conclusions we used that for $z_1 = z_2 = z$ it holds

$$\langle L_{\pm} R_{\pm} \rangle = |m|^2 \mp |z|^2 |u|^2, \quad (\text{B.5})$$

and Taylor expansion to access the $|z_1 - z_2| \ll 1$ regime.

We will also need to express the standard basis vectors I, E_- in terms of L_{\pm} . For $z_1 = z_2$ it holds $I = L_-, E_- = L_+$ if $\eta_1 \eta_2 < 0$, and $I = L_+, E_- = L_-$ if $\eta_1 \eta_2 > 0$. Then again using a simple Taylor expansion it follows that

$$I = (1 + \mathcal{O}(|z_1 - z_2|)) L_{\sigma} + \mathcal{O}(|z_1 - z_2|) L_{-\sigma}, \quad E_- = (1 + \mathcal{O}(|z_1 - z_2|)) L_{-\sigma} + \mathcal{O}(|z_1 - z_2|) L_{\sigma}, \quad (\text{B.6})$$

with $\sigma := \text{sign}(\eta_1 \eta_2)$.

REFERENCES

- ¹A. Adhikari and J. Huang, *Dyson Brownian motion for general β and potential at the edge*, *Probab. Theory Related Fields* **178**, 893–950 (2020).
- ²A. Adhikari and B. Landon, *Local law and rigidity for unitary Brownian motion*, preprint (2022), [arXiv:2202.06714](https://arxiv.org/abs/2202.06714).
- ³J. Alt, L. Erdős, and T. Krüger, *Local inhomogeneous circular law*, *Ann. Appl. Probab.* **28**, 148–203 (2018).
- ⁴J. Alt, L. Erdős, and T. Krüger, *Spectral radius of random matrices with independent entries*, *Probab. Math. Phys.* **2**, 221–280 (2021).
- ⁵Z. D. Bai, *Circular law*, *Ann. Probab.* **25**, 494–529 (1997).
- ⁶P. Bourgade and G. Dubach, *The distribution of overlaps between eigenvectors of Ginibre matrices*, *Probab. Theory Related Fields* **177**, 397–464 (2020).
- ⁷P. Bourgade, *Extreme gaps between eigenvalues of Wigner matrices*, *J. Eur. Math. Soc. (JEMS)* **24**, 2823–2873 (2022).
- ⁸P. Bourgade, L. Erdős, H.-T. Yau, and J. Yin, *Fixed energy universality for generalized Wigner matrices*, *Comm. Pure Appl. Math.* **69**, 1815–1881 (2016).
- ⁹P. Bourgade and H. Falconet, *Liouville quantum gravity from random matrix dynamics*, preprint (2022), [arXiv:2206.03029](https://arxiv.org/abs/2206.03029).

- ¹⁰P. Bourgade, H.-T. Yau, and J. Yin, *Local circular law for random matrices*, *Probab. Theory Related Fields* **159**, 545–595 (2014).
- ¹¹Z. Che and P. Lopatto, *Universality of the least singular value for sparse random matrices*, *Electron. J. Probab.* **24**, Paper No. 9, 53 (2019).
- ¹²G. Cipolloni, L. Erdős, and D. Schröder, *On the Condition Number of the Shifted Real Ginibre Ensemble*, *SIAM J. Matrix Anal. Appl.* **43**, 1469–1487 (2022).
- ¹³G. Cipolloni, L. Erdős, and D. Schröder, *Central limit theorem for linear eigenvalue statistics of non-Hermitian random matrices*, *Comm. Pure Appl. Math.* (2019), [arXiv:1912.04100](#).
- ¹⁴G. Cipolloni, L. Erdős, and D. Schröder, *Edge universality for non-Hermitian random matrices*, *Probab. Theory Related Fields* **179**, 1–28 (2021).
- ¹⁵G. Cipolloni, L. Erdős, and D. Schröder, *Eigenstate thermalization hypothesis for Wigner matrices*, *Comm. Math. Phys.* **388**, 1005–1048 (2021).
- ¹⁶G. Cipolloni, L. Erdős, and D. Schröder, *Fluctuation around the circular law for random matrices with real entries*, *Electron. J. Probab.* **26**, Paper No. 24, 61 (2021).
- ¹⁷G. Cipolloni, L. Erdős, and D. Schröder, *Functional central limit theorems for Wigner matrices*, Accepted for publication in *Ann. Appl. Probab* (2020), [arXiv:2012.13218](#).
- ¹⁸G. Cipolloni, L. Erdős, and D. Schröder, *Optimal lower bound on the least singular value of the shifted Ginibre ensemble*, *Probab. Math. Phys.* **1**, 101–146 (2020).
- ¹⁹G. Cipolloni, L. Erdős, and D. Schröder, *Optimal multi-resolvent local laws for Wigner matrices*, *Electron. J. Probab.* **27**, – (2022).
- ²⁰G. Cipolloni, L. Erdős, and D. Schröder, *Rank-uniform local law for Wigner matrices*, preprint (2022), [arXiv:2203.01861](#).
- ²¹G. Cipolloni, L. Erdős, and D. Schröder, *Thermalisation for Wigner matrices*, *J. Funct. Anal.* **282**, Paper No. 109394, 37 (2022).
- ²²N. Coston and S. O’Rourke, *Gaussian fluctuations for linear eigenvalue statistics of products of independent iid random matrices*, *J. Theoret. Probab.* **33**, 1541–1612 (2020).
- ²³L. Erdős, A. Knowles, H.-T. Yau, and J. Yin, *The local semicircle law for a general class of random matrices*, *Electron. J. Probab.* **18**, no. 59, 58 (2013).
- ²⁴L. Erdős, T. Krüger, and D. Schröder, *Random matrices with slow correlation decay*, *Forum Math. Sigma* **7**, e8, 89 (2019).
- ²⁵L. Erdős and H.-T. Yau, *Universality of local spectral statistics of random matrices*, *Bull. Amer. Math. Soc. (N.S.)* **49**, 377–414 (2012).
- ²⁶P. J. Forrester, *Fluctuation formula for complex random matrices*, *J. Phys. A* **32**, L159–L163 (1999).
- ²⁷Y. V. Fyodorov, *On statistics of bi-orthogonal eigenvectors in real and complex Ginibre ensembles: combining partial Schur decomposition with supersymmetry*, *Comm. Math. Phys.* **363**, 579–603 (2018).
- ²⁸V. L. Girko, *The circular law*, *Teor. Veroyatnost. i Primenen.* **29**, 669–679 (1984).
- ²⁹Y. He and A. Knowles, *Mesoscopic eigenvalue statistics of Wigner matrices*, *Ann. Appl. Probab.* **27**, 1510–1550 (2017).
- ³⁰J. Huang and B. Landon, *Rigidity and a mesoscopic central limit theorem for Dyson Brownian motion for general β and potentials*, *Probab. Theory Related Fields* **175**, 209–253 (2019).
- ³¹A. M. Khorunzhy, B. A. Khoruzhenko, and L. A. Pastur, *Asymptotic properties of large random matrices with independent entries*, *J. Math. Phys.* **37**, 5033–5060 (1996).
- ³²P. Kopel, *Linear statistics of non-Hermitian matrices matching the real or complex ginibre ensemble to four moments*, preprint (2015), [arXiv:1510.02987](#).
- ³³B. Landon, P. Lopatto, and P. Sosoe, *Single eigenvalue fluctuations of general Wigner-type matrices*, preprint (2021), [arXiv:2105.01178](#).
- ³⁴B. Landon and P. Sosoe, *Almost-optimal bulk regularity conditions in the clt for Wigner matrices*, preprint (2022), [arXiv:2204.03419](#).
- ³⁵B. Landon, P. Sosoe, and H.-T. Yau, *Fixed energy universality of Dyson Brownian motion*, *Adv. Math.* **346**, 1137–1332 (2019).
- ³⁶J. O. Lee and K. Schnelli, *Edge universality for deformed Wigner matrices*, *Rev. Math. Phys.* **27**, 1550018, 94 (2015).
- ³⁷J. Miller and S. Sheffield, *Imaginary geometry iv: interior rays, whole-plane reversibility, and space-filling trees*, preprint (2013), [arXiv:1302.4738](#).
- ³⁸H. H. Nguyen and V. Vu, *Random matrices: law of the determinant*, *Ann. Probab.* **42**, 146–167 (2014).
- ³⁹I. Nourdin and G. Peccati, *Universal Gaussian fluctuations of non-Hermitian matrix ensembles: from weak convergence to almost sure CLTs*, *ALEA Lat. Am. J. Probab. Math. Stat.* **7**, 341–375 (2010).

- ⁴⁰S. O'Rourke and D. Renfrew, *Central limit theorem for linear eigenvalue statistics of elliptic random matrices*, *J. Theoret. Probab.* **29**, 1121–1191 (2016).
- ⁴¹B. Rider, *Deviations from the circular law*, *Probab. Theory Related Fields* **130**, 337–367 (2004).
- ⁴²B. Rider and J. W. Silverstein, *Gaussian fluctuations for non-Hermitian random matrix ensembles*, *Ann. Probab.* **34**, 2118–2143 (2006).
- ⁴³B. Rider and B. Virág, *The noise in the circular law and the Gaussian free field*, *Int. Math. Res. Not. IMRN*, Art. ID rnm006, 33 (2007).
- ⁴⁴A. Sankar, D. A. Spielman, and S.-H. Teng, *Smoothed analysis of the condition numbers and growth factors of matrices*, *SIAM J. Matrix Anal. Appl.* **28**, 446–476 (2006).
- ⁴⁵M. Shcherbina and T. Shcherbina, *The least singular value of the general deformed Ginibre ensemble*, preprint (2022), [arXiv:2204.06026](https://arxiv.org/abs/2204.06026).
- ⁴⁶N. J. Simm, *Central limit theorems for the real eigenvalues of large Gaussian random matrices*, *Random Matrices Theory Appl.* **6**, 1750002, 18 (2017).
- ⁴⁷T. Tao and V. Vu, *Random matrices: the circular law*, *Commun. Contemp. Math.* **10**, 261–307 (2008).
- ⁴⁸T. Tao and V. Vu, *Random matrices: universality of local spectral statistics of non-Hermitian matrices*, *Ann. Probab.* **43**, 782–874 (2015).
- ⁴⁹T. Tao and V. Vu, *Smooth analysis of the condition number and the least singular value*, *Math. Comp.* **79**, 2333–2352 (2010).
- ⁵⁰K. Tikhomirov, *Invertibility via distance for noncentered random matrices with continuous distributions*, *Random Structures Algorithms* **57**, 526–562 (2020).