FLUCTUATIONS OF FUNCTIONS OF WIGNER MATRICES

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1. INTRODUCTION

The density of states of an $N \times N$ Wigner random matrix $H = H^{(N)}$ converges to the Wigner semicircular law [19]. More precisely, for any continuous function $f \colon \mathbb{R} \to \mathbb{C}$

$$\text{(1.1)} \qquad \lim_{N \to \infty} \frac{1}{N} \operatorname{Tr} f(H) = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} f(\lambda_k) = \int f(x) \, \mu_{sc}(\mathrm{d}x)$$

where $\lambda_1, \dots, \lambda_N$ are the (real) eigenvalues of H and $\mu_{sc}(\mathrm{d}x) := \frac{1}{2\pi} \sqrt{(4 - x^2)_+} \, \mathrm{d}x$.

It is well known that for regular functions f, the normalized linear eigenvalue statistics $\frac{1}{N} \operatorname{Tr} f(H)$ have an asymptotically Gaussian fluctuation on scale of order $1/N$, see, for example, [17, 8, 1, 15, 14, 6, 2] for different results in this direction, also for other random matrix ensembles. To our knowledge, this result under the weakest regularity condition on f was proved in [17]; for general Wigner matrices $f \in H^{1+\epsilon}$ was required, while for Wigner matrices with substantial GUE component $f \in H^{1/2+\epsilon}$ was sufficient. Notice that the order of the fluctuation $1/N$ is much smaller than $1/\sqrt{N}$ which would be predicted by the standard central limit theorem (CLT) if the eigenvalues were weakly dependent. The failure of CLT on scale $1/\sqrt{N}$ is a signature of the strong correlations among the eigenvalues.

In this paper we investigate the individual matrix elements of $f(H)$. We will show that the semicircle law (1.1) holds also for any diagonal matrix element $f(H)_{ii}$ and not only for their average, $\frac{1}{N} \operatorname{Tr} f(H)$; however, the corresponding fluctuation is much larger, it is on scale $1/\sqrt{N}$. Moreover, the limiting distribution of the rescaled fluctuation is not necessarily Gaussian; it also depends on the distribution of the matrix element h_{ii}. Similar fluctuation results hold for the off diagonal matrix elements $f(H)_{ij}$, $i \neq j$. For regularity condition, we merely assume that f is of bounded variation, $f \in BV$. We also prove an effective error bound of order $N^{-2/3}$ that we can improve to N^{-1} if $f' \in L^\infty$, i.e. we provide a two-term expansion for each matrix element of $f(H)$.

Similar results (with less precise error bounds) were obtained previously in [9] for Gaussian random matrices and in [10, 11, 12] for general Wigner matrices under the much stronger regularity assumptions that

$$\text{(1.2)} \qquad \int_{\mathbb{R}} (1 + |\xi|)^3 \left| \widehat{f}(\xi) \right| \mathrm{d}\xi < \infty \quad \text{or} \quad \int_{\mathbb{R}} (1 + |\xi|)^{2s} \left| \widehat{f}(\xi) \right|^2 \mathrm{d}\xi < \infty \quad \text{for some } s > 3,$$

where $\widehat{f}(\xi) := \int_{\mathbb{R}} e^{-i\xi x} f(x) \, \mathrm{d}x$. The main novelty of the current work is thus to relax these regularity conditions to $f \in BV$. In addition, [10, 11, 12] assumed that in the case of complex Hermitian matrices, the real and imaginary part of the entries have equal variance. Our approach does not require this technical assumption. We also refer to [7] where similar questions have been studied for more general statistics of the form $\operatorname{Tr}[f(H)A]$ for non-random matrices A under the fairly strong regularity condition $\int (1 + |\xi|)^4 |\widehat{f}(\xi)| \, \mathrm{d}\xi < \infty$.

A special case of these questions is when the test function $f(x)$ is given by $\varphi_z(x) = (x - z)^{-1}$ for some complex parameter z in the upper half plane, $\eta := \Im z > 0$. In fact, for f which are analytic in a complex neighborhood of $[-2, 2]$, a simple contour integration shows that for the linear statistics it is sufficient to understand the resolvent of H, i.e., $\varphi_z(H) = (H - z)^{-1}$ for any fixed z in the upper half plane. If f is less regular, one may still express $f(H)$ as an integral of the resolvents over z, weighted by the $\partial_{\bar{z}}$-derivative of an almost analytic extension of f to the upper half plane (Helffer-Sjöstrand formula). In this case, the integration effectively involves the regime of z close to the real axis, so the resolvent $(H - z)^{-1}$ and its matrix elements need to be controlled even as $\eta \to 0$ simultaneously with $N \to \infty$. These results are commonly called *local semicircle laws*. They hold down to the optimal scale $\eta \gg 1/N$ with an optimal error bound of order $1/\sqrt{N\eta}$ for the individual matrix elements and a bound of order $1/N\eta$ for the normalized trace of the resolvent (see, e.g. [5]). With the help of the Helffer-Sjöstrand formula, more accurate local laws can be transformed to weaker regularity assumptions on the test function in the linear eigenvalue statistics, see [17]. In this paper we replace the Helffer-Sjöstrand formula by Pleijel's formula [13] that provides a more effective functional calculus for functions with low regularity.

A similar relation between regularity and local laws holds for individual matrix elements, $f(H)_{ii}$. Using the Schur complement formula one can relate $f(H)_{ii}$ to the *difference* of a linear statistics for H and for its minor $\widehat{H}$ obtained by removing the i-th row and column from H. In a recent paper [4] we investigated the fluctuations of this difference without directly connecting it to $f(H)_{ii}$. Applied to a special family of test function $f(x) = |x - a|$, the difference of linear statistics is closely related to the fluctuation of Kerov's interlacing sequences of the eigenvalues of H and its minor.

Motivated by this application, Sasha Sodin pointed out that this fluctuation can be related to the fluctuation of a single matrix element of the resolvent by the Markov correspondence, see [16] for details. It is therefore natural to ask if one could use the fluctuation result from [4] on the interlacing sequences to strengthen the existing results on the fluctuations of the matrix elements of the resolvent and hence of $f(H)$. In fact, not the result itself, but the core of the analysis in [4] can be applied; this is the content of the current paper. We thank Sasha for asking this question and calling our attention to the problem of fluctuation of the matrix elements of $f(H)$ and to the previous literature [9, 10, 11, 12]. Furthermore, he pointed out to us that the contour integral formula from Pleijel's paper [13] could potentially replace the Helffer-Sjöstrand formula in our argument to the end of further reducing the regularity assumptions on f. We are very grateful to him for this insightful idea that we believe will have further applications.

2. Main results

We consider complex Hermitian and real symmetric random $N \times N$ matrices $H = (h_{ij})_{i,j=1}^N$ with the entries being independent (up to the symmetry constraint $h_{ij} = \overline{h_{ji}}$) random variables satisfying

$$(2.1) \qquad \mathbf{E}\, h_{ij} = 0, \quad \mathbf{E}\, |h_{ij}|^2 = \frac{s_{ij}}{N} \quad \text{and} \quad \mathbf{E}\, |h_{ij}|^p \le \frac{\mu_p}{N^{p/2}}$$

for all i, j, p and some absolute constants μ_p. We assume that the matrix of variances is approximately stochastic, i.e.

$$(2.2) \qquad \sum_j s_{ij} = N + \mathcal{O}(1)$$

to guarantee that the limiting density of states is the Wigner semicircular law.

To formulate the error bound concisely we introduce the following commonly used (see, e.g., [3]) notion of high probability bound.

Definition 2.1 (Stochastic Domination)**.** *If*

$$X = \left(X^{(N)}(u) \,|\, N \in \mathbb{N}, u \in U^{(N)} \right) \quad \text{and} \quad Y = \left(Y^{(N)}(u) \,|\, N \in \mathbb{N}, u \in U^{(N)} \right)$$

are families of random variables indexed by N, and possibly some parameter u, then we say that X is stochastically dominated by Y, if for all $\epsilon, D > 0$ we have

$$\sup_{u \in U^{(N)}} \mathbf{P}\left[X^{(N)}(u) > N^\epsilon Y^{(N)}(u) \right] \le N^{-D}$$

for large enough $N \geq N_0(\epsilon, D)$. In this case we use the notation $X \prec Y$. Moreover, if we have $|X| \prec Y$, we also write $X = \mathcal{O}_\prec(Y)$.

It can be checked (see [3, Lemma 4.4]) that $\prec$ satisfies the usual arithmetic properties, e.g. if $X_1 \prec Y_1$ and $X_2 \prec Y_2$, then also $X_1 + X_2 \prec Y_1 + Y_2$ and $X_1 X_2 \prec Y_1 Y_2$. We will say that a (sequence of) events $A = A^{(N)}$ holds with *overwhelming probability* if $\mathbf{P}(A^{(N)}) \geq 1 - N^{-D}$ for any $D > 0$ and $N \geq N_0(D)$. In particular, under the conditions (2.1), we have $h_{ij} \prec N^{-1/2}$ and $\max_k |\lambda_k| \leq 3$ with overwhelming probability.

We further introduce a notion quantifying the rate of weak convergence of distributions. We say that a sequence of random variables X_N *converges in distribution at a rate $r(N)$* to X if for any $t \in \mathbb{R}$ it holds that

$$\mathbf{E}\, e^{itX_N} = \mathbf{E}^{itX} + \mathcal{O}_t(r(N)),$$

where we allow the coefficient of the rate to be t-dependent uniformly for $|t| \leq T$ for any fixed T. If X_N converges in distribution at a rate $r(N)$, we write

$$X_N \overset{d}{=} X + \mathcal{O}(r(N)).$$

In particular, this implies that

$$\mathbf{E}\, \Phi(X_N) = \mathbf{E}\, \Phi(X) + \mathcal{O}(r(N))$$

for any analytic function Φ with compactly supported Fourier transform.

Our main result for the diagonal entries of $f(H)$ is summarized in the following theorem. By permutational symmetry there is no loss in generality in studying $f(H)_{11}$. By considering real and imaginary parts separately, from now on we always assume that f is real valued.

Theorem 2.2. *Let the Wigner matrix H satisfy (2.1), $s_{ij} = 1$ for $i \neq j$ and $s_{ii} \leq C$ for all i, $\mathbf{E}\,|h_{1j}|^4 = \sigma_4/N^2$ for $j = 2, \ldots, N$ and $\mathbf{E}\, h_{ij}^2 = \sigma_2/N$ with some $\sigma_2, \sigma_4 \in \mathbb{R}$. Moreover, let $f \in BV([-3,3])$ be some real-valued function of bounded variation and assume that $h_{11} \overset{d}{=} \xi_{11}/\sqrt{N}$ where ξ_{11} is an N-independent random variable. Then*

$$(2.3) \quad f(H)_{11} \overset{d}{=} \int f(x)\, \mu_{sc}(\mathrm{d}x) + \frac{\widehat{\Delta}_f + \xi_{11} \int f(x)x\, \mu_{sc}(\mathrm{d}x)}{\sqrt{N}} + \begin{cases} \mathcal{O}(N^{-1}) & \text{if } f' \in L^\infty, \\ \mathcal{O}(N^{-2/3}) & \text{else,} \end{cases}$$

where $\widehat{\Delta}_f$ is a centered Gaussian random variable of variance

$$(2.4) \quad \mathbf{E}\left(\widehat{\Delta}_f\right)^2 = V_{f,1} + V_{f,1}^{(\sigma_2)} - 2V_{f,2} - (1+\sigma_2)V_{f,3} + (\sigma_4 - 2 - \sigma_2^2)V_{f,4},$$

and the $V_{f,i}$ and $V_{f,1}^{(\sigma_2)}$ are given by quadratic forms defined in (4.13).

More precisely, (2.3) means that, to leading order

$$(2.5) \quad f(H)_{11} = \int f(x)\, \mu_{sc}(\mathrm{d}x) + \mathcal{O}_\prec\left(N^{-1/2}\right)$$

and, weakly

$$(2.6) \quad T_f^{(N)} := \sqrt{N}\left[f(H)_{11} - \int f(x)\, \mu_{sc}(\mathrm{d}x)\right] - \xi_{11}\int f(x)x\, \mu_{sc}(\mathrm{d}x) \Rightarrow \widehat{\Delta}_f$$

at a speed

$$\mathbf{E}\left(T_f^{(N)}\right)^k = \mathbf{E}\,\widehat{\Delta}_f^k + \begin{cases} \mathcal{O}\left(\dfrac{C^k(k/2)!}{\sqrt{N}}\right) & \text{if } f' \in L^\infty, \\ \mathcal{O}\left(\dfrac{C^k(k/2)!}{N^{1/6}}\right) & \text{else} \end{cases}$$

for all k. The speed of convergence in the Lévy metric d_L is given by

$$(2.7) \quad d_L(T_f^{(N)}, \widehat{\Delta}_f) \leq C(f)\frac{\log \log N}{\sqrt{\log N}}$$

with some constant depending on f.

The corresponding result for the off diagonal terms is as follows.

Theorem 2.3. *Under the assumptions of Theorem 2.2,*

$$(2.8) \qquad f(H)_{12} \overset{d}{=} \frac{1}{\sqrt{N}} \left[\widetilde{\Delta}_f + \xi_{12} \int f(x)x \, \mu_{sc}(\mathrm{d}x) \right] + \begin{cases} \mathcal{O}\left(N^{-1}\right) & \text{if } f' \in L^\infty, \\ \mathcal{O}\left(N^{-2/3}\right) & \text{else,} \end{cases}$$

where $\widetilde{\Delta}_f$ is a centered complex Gaussian satisfying

$$\mathbf{E}\,\widetilde{\Delta}_f^2 = V_{f,1}^{(\sigma_2)} - V_{f,2} - \sigma_2 V_{f,3}, \quad \mathbf{E}\left|\widetilde{\Delta}_f\right|^2 = V_{f,1} - V_{f,2} - V_{f,3}.$$

and the $V_{f,i}$ and $V_{f,1}^{(\sigma_2)}$ are defined in (4.13).

More precisely, (2.8) means that

$$(2.9) \qquad\qquad\qquad f(H)_{12} = \mathcal{O}_\prec\left(N^{-1/2}\right)$$

and, introducing the notation

$$S_f^{(N)} := \sqrt{N} f(H)_{12} - \xi_{12} \int f(x)x \, \mu_{sc}(\mathrm{d}x),$$

we have that

$$\mathbf{E}\left(S_f^{(N)}\right)^k \overline{\left(S_f^{(N)}\right)}^l = \mathbf{E}\,\widetilde{\Delta}_f^k \overline{\widetilde{\Delta}_f}^l + \begin{cases} \mathcal{O}\left(\frac{((k+l)/2)!}{\sqrt{N}}\right) & \text{if } f' \in L^\infty, \\ \mathcal{O}\left(\frac{((k+l)/2)!}{N^{1/6}}\right) & \text{else} \end{cases}$$

holds for all $k, l \in \mathbb{N}$. The analogues of (2.6) and (2.7) also hold for $T_f^{(N)}$ replaced with $S_f^{(N)}$.

The fluctuation results in Theorems 2.2 and 2.3 for test functions satisfying the stronger regularity assumption (1.2) and without explicit error terms have been proven in [10, 11].

We also remark that (2.6) implies the joint asymptotic normality of the fluctuations of $f(H^{(N)})_{11}$ for several test functions. More precisely, for any $f \in BV$ we define $T_f^{(N)}$ via (2.6). Then for any given functions $f_1, f_2, \ldots, f_k \in BV$, the random k-vector

$$\left(T_{f_1}^{(N)}, T_{f_2}^{(N)}, \ldots, T_{f_k}^{(N)}\right)$$

weakly converges to a Gaussian vector with covariance given via the variance (2.4) using the parallelogram identity. Similar result holds for the joint distribution of the off diagonal elements $f_k(H)_{12}$. One may specialize this result to the case when f is a characteristic function, i.e. we may define

$$T_x^{(N)} := T_{\mathbf{1}_{[-3,x]}}^{(N)}, \qquad x \in [-3, 3],$$

where $\mathbf{1}_{[a,b]}$ is the characteristic function of the interval $[a, b]$. Clearly, the finite dimensional marginals of the sequence of stochastic processes $\{T_x^{(N)}, x \in [-3, 3]\}$ are asymptotically Gaussian. The tightness remains an open question.

3. Pleijel's inversion formula

Our main tool relating $f(H)_{ij}$ to the resolvent $G = G(z) = (H - z)^{-1}$ is summarized in the following proposition. We formulate it for general probability measures μ supported on some $[-K, K]$ and their Stieltjes transform

$$m_\mu(z) = \int \frac{1}{\lambda - z} \mu(\mathrm{d}\lambda).$$

Later we will apply the proposition to $\mu = \rho_N$ and $\mu = \widetilde{\rho}_N$ with $\rho_N, \widetilde{\rho}_N$ being the spectral measures of typical diagonal and off-diagonal entries

$$\int f \, \mathrm{d}\rho_N = f(H)_{11}, \quad \int f \, \mathrm{d}\widetilde{\rho}_N = f(H)_{12}.$$

Proposition 3.1. *Let $L > K > 0$ and let μ denote a probability measure which is supported on $[-K, K]$ and let $f \in BV([-L, L])$ be a function of bounded variation which is compactly supported in $[-L, L]$. Then*

(3.1)
$$
\int f(\lambda)\,\mu(\mathrm{d}\lambda) = \frac{1}{2\pi} \iint_{I_{\eta_0}^M} m_\mu(x + i\eta)\,\mathrm{d}\eta\,\mathrm{d}f(x) + \frac{1}{\pi} \int_{-L}^{L} f(x)\Im m_\mu(x + Mi)\,\mathrm{d}x
$$
$$
+ \mathcal{O}\left(\eta_0\,\|m_\mu(\cdot + i\eta_0)\|_{L^1(|\mathrm{d}f|)}\right)
$$
$$
= \frac{1}{2\pi} \iint_{I_{\eta_0}^M} m_\mu(x + i\eta)\,\mathrm{d}\eta\,\mathrm{d}f(x) + \mathcal{O}\left(\eta_0\,\|m_\mu(\cdot + i\eta_0)\|_{L^1(|\mathrm{d}f|)} + \frac{1}{M}\,\|f\|_1\right)
$$

holds for any $\eta_0, M > 0$ where $I_{\eta_0}^M := [-L, L] \times ([-M, M] \setminus [-\eta_0, \eta_0])$, $\|\cdot\|_1 = \|\cdot\|_{L^1(\mathrm{d}x)}$ and $\mathrm{d}f$ is understood as the (signed) Lebesgue–Stieltjes measure.

Before going into the proof, we present a special case of Proposition 3.1. If $f = \mathbf{1}_{[x,x']}$, then (3.1) can be written as the path integral

(3.2)
$$
\mu([x, x']) = \frac{1}{2\pi i} \int_{\gamma(x,x')} m_\mu(z)\,\mathrm{d}z + \mathcal{O}\left(\eta_0[|m_\mu(x + i\eta_0)| + |m_\mu(x' + i\eta_0)|]\right),
$$

where $\gamma(x, x')$ is the chain indicated in Figure (1c). We also want to remark that for our purposes (3.1) is favorable over the Helffer-Sjöstrand representation, as used in [4], since it requires considerably less regularity on f.

Proof of Proposition 3.1. From [13, Eq. (5)] we know that

(3.3)
$$
\mu([-K, x)) = \frac{1}{2\pi i} \int_{L(x)} m_\mu(z)\,\mathrm{d}z + \frac{\eta_0}{\pi}\Re m_\mu(z_0) + \mathcal{O}\left(\eta_0 \Im m_\mu(z_0)\right),
$$

where $L(x)$ is a directed path as indicated in Figure 1a and $z_0 = x + i\eta_0$, $\eta_0 > 0$.

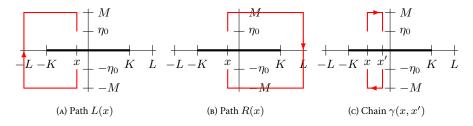(A) Path $L(x)$ (B) Path $R(x)$ (c) Chain $\gamma(x, x')$

FIGURE 1. Integration paths

By the definition of the Lebesgue–Stieltjes integral for functions of bounded variation we have that

$$
\int f(\lambda)\,\mu(\mathrm{d}\lambda) = \int_{-L}^{L} \left(\int \mathbf{1}(\lambda \geq x)\,\mu(\mathrm{d}\lambda)\right)\mathrm{d}f(x) = \int_{-L}^{L} \mu([x, K])\,\mathrm{d}f(x).
$$

By virtue of (3.3) we can write

$$
\int f(\lambda)\,\mu(\mathrm{d}\lambda) = \frac{1}{\pi} \int_{-L}^{L} \left(\frac{1}{2i} \int_{R(x)} m_\mu(z)\,\mathrm{d}z\right)\mathrm{d}f(x) + \mathcal{O}\left(\eta_0\,\|m_\mu(\cdot + i\eta_0)\|_{L^1(|\mathrm{d}f|)}\right),
$$

where $R(x)$ is the path indicated in Figure 1b and $|\mathrm{d}f|$ indicates the total variation measure of $\mathrm{d}f$. We then write out the inner integral as

$$
\frac{1}{2i} \int_{R(x)} m_\mu(z)\,\mathrm{d}z = \int_{\eta_0}^{M} \Re m_\mu(x + i\eta)\,\mathrm{d}\eta + \int_{x}^{L} \Im m_\mu(y + iM)\,\mathrm{d}y - \int_{0}^{M} \Re m_\mu(L + i\eta)\,\mathrm{d}\eta.
$$

Since the last term is x-independent, it will vanish after integrating against $\mathrm{d}f$ since we assumed f to be compactly supported. For the second term we find

$$\int f(\lambda)\,\mu(\mathrm{d}\lambda) = \frac{1}{\pi}\int_{-L}^{L}\int_{\eta_0}^{M}\Re m_\mu(x+i\eta)\,\mathrm{d}\eta\,\mathrm{d}f(x) + \frac{1}{\pi}\int_{-L}^{L}f(x)\Im m_\mu(x+iM)\,\mathrm{d}x$$
$$+ \mathcal{O}\left(\eta_0\,\|m_\mu(\cdot+i\eta_0)\|_{L^1(|\mathrm{d}f|)}\right).$$

Since $|\Im m_\mu(x+iM)| \leq 1/M$ we thus have

$$\int f(\lambda)\,\mu(\mathrm{d}\lambda) = \frac{1}{\pi}\int_{-L}^{L}\int_{\eta_0}^{M}\Re m_\mu(x+i\eta)\,\mathrm{d}\eta\,\mathrm{d}f(x) + \mathcal{O}\left(\eta_0\,\|m_\mu(\cdot+i\eta_0)\|_{L^1(|\mathrm{d}f|)} + \frac{1}{M}\|f\|_1\right)$$

for any $\eta_0, M > 0$. For applications it turns out to be favorable to get rid of the real part which we can by noting that $2\Re m_\mu(z) = m_\mu(z) + m_\mu(\overline{z})$ and therefore

$$\int f(\lambda)\,\mu(\mathrm{d}\lambda) = \frac{1}{2\pi}\iint_{I_{\eta_0}^M} m_\mu(x+i\eta)\,\mathrm{d}\eta\,\mathrm{d}f(x) + \mathcal{O}\left(\eta_0\,\|m_\mu(\cdot+i\eta_0)\|_{L^1(|\mathrm{d}f|)} + \frac{1}{M}\|f\|_1\right),$$

where we recall $I_{\eta_0}^M = [-L, L] \times ([-M, M] \setminus [-\eta_0, \eta_0])$.   □

We finally note that a variant of Proposition 3.1 could also be proven directly without appealing to the contour integration from [13]. The key computation in that direction is summarized in the following Lemma which we establish here for later convenience.

Lemma 3.2. *Let $f \in BV([-L, L])$ be compactly supported and let g be a function which is analytic away from the real axis and satisfies $g(\overline{z}) = \overline{g(z)}$. Then for any $\eta_0, M > 0$ we have that*

$$\frac{1}{2\pi}\iint_{I_{\eta_0}^M} g(x+i\eta)\,\mathrm{d}\eta\,\mathrm{d}f(x) = \frac{1}{\pi}\int_{-L}^{L}f(x)\Im g(x+i\eta_0)\,\mathrm{d}x + \mathcal{O}\left(\|f\|_1\max_{x\in[-L,L]}|g(x+iM)|\right).$$

Applying Lemma 3.2 to $g = m_\mu$ yields, modulo an error term,

$$\frac{1}{2\pi}\iint_{I_{\eta_0}^M} m_\mu(x+i\eta)\,\mathrm{d}\eta\,\mathrm{d}f(x) \approx \int\int_{-L}^{L}f(x)\frac{1}{\pi}\frac{\eta_0}{(\lambda-x)+\eta_0^2}\,\mathrm{d}x\,\mu(\mathrm{d}\lambda)$$

and taking the limit $\eta_0 \to 0$ makes the inner integral tend to $f(\lambda)$ in L^1-sense. In this way we can establish a variant of Proposition 3.1, albeit with a weaker error estimate.

Proof of Lemma 3.2. This follows from the computation

$$\iint_{I_{\eta_0}^M} g(x+i\eta)\,\mathrm{d}\eta\,\mathrm{d}f(x) = -i\int_{\partial I_{\eta_0}^M}f(x)g(z)\,\mathrm{d}z = 2\int_{-3}^{3}f(x)\Im\left[g(x+i\eta_0) - g(x+iM)\right]\mathrm{d}x$$

$$= 2\int_{-3}^{3}f(x)\Im g(x+i\eta_0)\,\mathrm{d}x + \mathcal{O}\left(\|f\|_1\max_{x\in[-3,3]}|g(x+iM)|\right),$$

where the first step follows from Stokes' or Green's Theorem.   □

4. Diagonal entries

We first prove Theorem 2.2 about the diagonal entries of $f(H)$. The spectral measure corresponding to the $(1,1)$-matrix element, ρ_N defined as

$$\int f\,\mathrm{d}\rho_N = f(H)_{11}$$

is concentrated in $[-2.5, 2.5]$ with overwhelming probability. We can without loss of generality assume that f is compactly supported in $[-3, 3]$ since smoothly cutting off f outside the spectrum does not change the result. Applying Proposition 3.1 to $\mu = \rho_N$ with $K = 2.5$, $L = 3$, we find that (using $z = x+i\eta$, $z_0 = x+i\eta_0$)

$$(4.1)\qquad f(H)_{11} = \frac{1}{2\pi}\iint_{I_{\eta_0}^M} G(z)_{11}\,\mathrm{d}\eta\,\mathrm{d}f(x) + \mathcal{O}_\prec\left(\eta_0\int|G(z_0)_{11}|\,\mathrm{d}f(x) + \frac{1}{M}\|f\|_1\right).$$

To analyse $G(z)_{11}$ we recall the Schur complement formula

$$G(z)_{11} = \frac{1}{h_{11} - z - \left\langle h, \widehat{G}(z)h \right\rangle}, \quad \text{where } H = \begin{pmatrix} h_{11} & h^* \\ h & \widehat{H} \end{pmatrix}, \qquad \widehat{G}(z) := (\widehat{H} - z)^{-1}.$$

To study the asymptotic behavior of $G(z)_{11}$ we rely on the local semicircle law in the averaged form (see [5] or [3, Theorem 2.3]) applied to the resolvent of the minor

$$\widehat{m}_N(z) = \frac{1}{N} \operatorname{Tr} \widehat{G}(z) = m(z) + \mathcal{O}_\prec \left(\frac{1}{N |\eta|} \right), \tag{4.2}$$

and its entry-wise form

$$G(z)_{ij} - \delta_{ij} m(z) \prec \frac{1}{\sqrt{N |\eta|}} \tag{4.3}$$

which both hold true for all $|\eta| = |\Im z| > \eta_0 \gg N^{-1}$. Here m denotes the Stieltjes transform of the semicircular distribution μ_{sc}, $m(z) := \int (\lambda - z)^{-1} \mu_{sc}(\mathrm{d}\lambda)$.

Since by (4.3),

$$\int |G(x + i\eta_0)_{11}| \, \mathrm{d} f(x) = \int |m(x + i\eta_0)| \, \mathrm{d} f(x) + \mathcal{O}_\prec \left(\int \left| \frac{1}{\sqrt{N}\eta_0} \right| \mathrm{d} f(x) \right) \prec \|\mathrm{d} f\|$$

for $\eta_0 \gg 1/N$, where $\|\mathrm{d} f\|$ is the total variation norm of the Lebesgue–Stieltjes measure $\mathrm{d} f$, we can write (4.1) as

$$f(H)_{11} = \frac{1}{2\pi} \iint_{I_{\eta_0}^M} G(x + i\eta)_{11} \, \mathrm{d}\eta \, \mathrm{d} f(x) + \mathcal{O}_\prec \left(\eta_0 \|\mathrm{d} f\| + M^{-1} \|f\|_1 \right).$$

In order to separate the leading order contribution from the fluctuation, we set

$$\Phi_N(z) = G(z)_{11} = \frac{1}{h_{11} - z - \left\langle h, \widehat{G}(z)h \right\rangle}, \qquad \widehat{\Phi}_N(z) = \frac{1}{-z - \widehat{m}_N(z)},$$

where $\widehat{m}_N(z) = \frac{1}{N} \operatorname{Tr} \widehat{G}(z)$ and observe that

$$\widehat{\Phi}_N(z) = \frac{1}{-z - m(z)} + \frac{\mathcal{O}_\prec (m(z) - \widehat{m}_N(z))}{-z - m(z)} = m(z) + \mathcal{O}_\prec \left(\frac{1}{N |\eta|} \right) \tag{4.4}$$

and by expanding both terms around $[-z - m(z)]^{-1} = m(z)$,

$$\Phi_N(z) - \widehat{\Phi}_N(z) = m(z)^2 \left[\left\langle h, \widehat{G}(z)h \right\rangle - \widehat{m}_N(z) - h_{11} \right] + \mathcal{O}_\prec \left(\frac{1}{N |\eta|} \right). \tag{4.5}$$

Thus $\widehat{\Phi}_N$ describes the leading order behavior, which is very close to a deterministic quantity, and the leading fluctuation is solely described by $\Phi_N - \widehat{\Phi}_N$. We then can write

$$f(H)_{11} = \Lambda_f^{(N)} + \frac{\Delta_f^{(N)}}{\sqrt{N}} + \mathcal{O}_\prec \left(\eta_0 \|\mathrm{d} f\| + \frac{1}{M} \|f\|_1 \right),$$

where

$$\Lambda_f^{(N)} := \frac{1}{2\pi} \iint_{I_{\eta_0}^M} \widehat{\Phi}_N(z) \, \mathrm{d}\eta \, \mathrm{d} f(x) \quad \text{and} \quad \Delta_f^{(N)} := \frac{1}{2\pi} \iint_{I_{\eta_0}^M} \sqrt{N} [\Phi_N - \widehat{\Phi}_N(z)] \, \mathrm{d}\eta \, \mathrm{d} f(x).$$

The reason for the normalization will become apparent later since in this way $\Delta_f^{(N)}$ is an object of order 1.

For the leading order term we use (4.4) and Proposition 3.1 to compute

$$\Lambda_f^{(N)} = \frac{1}{2\pi} \int_{I_{\eta_0}^M} m(z) \, \mathrm{d}\eta \, \mathrm{d} f(x) + \mathcal{O}_\prec \left(\|\mathrm{d} f\| \int_{\eta_0}^M \frac{1}{N\eta} \, \mathrm{d}\eta \right)$$

$$= \int f(x) \, \mu_{sc}(\mathrm{d}x) + \mathcal{O}_\prec \left(\left[\frac{|\log M| + |\log \eta_0|}{N} + \eta_0 \right] \|\mathrm{d} f\| + \frac{1}{M} \|f\|_1 \right).$$

For the fluctuation we use (4.5) to compute

$$\Delta_f^{(N)} = \frac{1}{2\pi} \int_{I_{\eta_0}^M} m(z)^2 \sqrt{N} \left[\left\langle h, \widehat{G}(z)h \right\rangle - \widehat{m}_N(z) - h_{11} \right] \mathrm{d}\eta \, \mathrm{d}f(x)$$

$$+ \mathcal{O}_\prec \left(\frac{|\log M| + |\log \eta|}{\sqrt{N}} \|\mathrm{d}f\| \right)$$

$$(4.6) \qquad = \widehat{\Delta}_f^{(N)} - \xi_{11} \frac{1}{2\pi} \int_{I_{\eta_0}^M} m(z)^2 \, \mathrm{d}\eta \, \mathrm{d}f(x) + \mathcal{O}_\prec \left(\frac{|\log M| + |\log \eta|}{\sqrt{N}} \|\mathrm{d}f\| \right)$$

$$= \widehat{\Delta}_f^{(N)} + \xi_{11} \int f(x) x \, \mu_{sc}(\mathrm{d}x) + \mathcal{O}_\prec \left(\frac{|\log M| + |\log \eta|}{\sqrt{N}} \|\mathrm{d}f\| + \eta_0 + \frac{1}{M^2} \|f\|_1 \right),$$

where the last step followed from Lemma 3.2 and

$$\xi_{11} = \sqrt{N} h_{11}, \quad \widehat{\Delta}_f^{(N)} := \frac{1}{2\pi} \int_{I_{\eta_0}^M} m(z)^2 X(z) \, \mathrm{d}\eta \, \mathrm{d}f(x), \quad X(z) = \sqrt{N} \left[\left\langle h, \widehat{G}(z)h \right\rangle - \widehat{m}_N(z) \right].$$

We now concentrate on the computation of $\mathbf{E} \left(\widehat{\Delta}_f^{(N)} \right)^2$. We state the main estimate of $\mathbf{E} \, X(z) X(z')$ as a lemma.

Lemma 4.1. *Under the assumptions of Theorem 2.2 it holds that*

(4.7)
$$\mathbf{E} \, X(z) X(z') = \frac{m(z)^2 m(z')^2}{1 - m(z)m(z')} + \frac{\sigma_2^3 m(z)^2 m(z')^2}{1 - \sigma_2 m(z)m(z')} + (\sigma_4 - 1)m(z)m(z') + \mathcal{O}_\prec \left(\frac{\Psi}{\sqrt{N}\Phi} \right),$$

where

$$\Psi := \frac{1}{\sqrt{|\eta\eta'|}} \left(\frac{1}{\sqrt{|\eta|}} + \frac{1}{\sqrt{|\eta'|}} + \frac{1}{\sqrt{N}|\eta\eta'|} \right)$$

$$\Phi := \mathbf{1}_{|x|,|x'| \leq 2} \left(|\eta| + |\eta'| + |x - x'|^2 \right) + \left[(|x| - 2)_+ + (|x'| - 2)_+ \right]$$

and $z = x + i\eta$, $z' = x' + i\eta'$.

We remark that in the $|x - x'|^2$ term in Φ could be replaced by $|x - x'|$ but we will not need this stronger bound here.

Proof of Lemma 4.1. From (35) in [4] we know that

$$(4.8) \qquad \mathbf{E} \left[X(z) X(z') | \widehat{H} \right] = \frac{1}{N} \sum_{i \neq j} \left(\widehat{G}_{ij} \widehat{G}'_{ji} + \sigma_2^2 \widehat{G}_{ij} \widehat{G}'_{ji} \right) + \frac{\sigma_4 - 1}{N} \sum_i \widehat{G}_{ii} \widehat{G}'_{ii}$$

where, $\widehat{G}_{ij} := \widehat{G}(z)_{ij}$, $\widehat{G}'_{ij} := \widehat{G}(z')_{ij}$. The last term we directly estimate as

$$(4.9) \quad \frac{\sigma_4 - 1}{N} \sum_i \widehat{G}_{ii} \widehat{G}'_{ii} = (\sigma_4 - 1)m(z)m(z') + \mathcal{O}_\prec \left(\frac{1}{\sqrt{N|\eta|}} + \frac{1}{\sqrt{N|\eta'|}} + \frac{1}{N\sqrt{|\eta\eta'|}} \right).$$

Furthermore, in Lemma 9 of [4] self-consistent equations for the first two terms on the rhs. of (4.8) were derived. We recall that

$$[1 - m(z)m(z')] \frac{1}{N} \sum_{i \neq j} \widehat{G}_{ij} \widehat{G}'_{ji} = m(z)^2 m(z')^2 + \mathcal{O}_\prec \left(\frac{\Psi}{\sqrt{N}} \right),$$

$$[1 - \sigma_2 m(z)m(z')] \frac{1}{N} \sum_{i \neq j} \widehat{G}_{ij} \widehat{G}'_{ij} = \sigma_2 m(z)^2 m(z')^2 + \mathcal{O}_\prec \left(\frac{\Psi}{\sqrt{N}} \right),$$

Using the straightforward inequality $|m(z)| \leq 1 - c|\eta|$, which holds for some small $c > 0$ and z in the compact region $[-10, 10] \times [-i, i]$, we find

$$|1 - m(z)m(z')| \geq c(|\eta| + |\eta'|).$$

Since $|m|$ decays outside the spectrum $[-2, 2]$ we have that $|m(z)| \leq 1 - c'(|x| - 2)_+$ for $|z| \leq 10$, and therefore

$$\left|1 - m(z)m(z')\right| \geq c'(|x| - 2)_+ + c'(|x'| - 2)_+.$$

Moreover, in the remaining regime where both $|\eta|, |\eta'| \ll 1$ and $|x|, |x'| \leq 2$, it holds that

$$\left|1 - m(z)m(z')\right| \geq 1 - \Re[m(z)m(z')] = 1 - (\Re m(z))(\Re m(z')) + (\Im m(z))(\Im m(z'))$$

$$\geq c''\left(1 - \frac{xx'}{4} \pm \frac{\sqrt{4 - x^2}\sqrt{4 - x'^2}}{4}\right) \geq c''(x - x')^2,$$

where the $\pm$ depends on the signs of η, η' and we allow for the constant c'' to change in the last inequality. This estimate follows from the explicit formula for $m(z)$. Putting these inequalities together, we therefore find a constant $C > 0$ such that in the compact region $[-3, 3] \times [-iM, iM]$ it holds that $C|1 - m(z)m(z')| \geq \Phi$, from which we obtain

$$(4.10) \qquad \frac{1}{N} \sum_{i \neq j} \widehat{G}_{ij} \widehat{G}'_{ji} = \frac{m(z)^2 m(z')^2}{1 - m(z)m(z')} + \mathcal{O}_\prec\left(\frac{\Psi}{\sqrt{N}\Phi}\right),$$

$$\frac{1}{N} \sum_{i \neq j} \widehat{G}_{ij} \widehat{G}'_{ij} = \frac{\sigma_2 m(z)^2 m(z')^2}{1 - \sigma_2 m(z)m(z')} + \mathcal{O}_\prec\left(\frac{\Psi}{\sqrt{N}\Phi}\right).$$

Now (4.7) follows from combining (4.8), (4.9) and (4.10). $\qquad \square$

Using Lemma 4.1 we then compute

$$\mathbf{E}\left(\widehat{\Delta}_f^{(N)}\right)^2 = \frac{1}{(2\pi)^2} \iiiint_{I_{\eta_0}^M} m(z)^2 m(z')^2 \, \mathbf{E}\, X(z)X(z') \, \mathrm{d}\boldsymbol{\eta} \, \mathrm{d}f(\boldsymbol{x})$$

$$= \frac{1}{(2\pi)^2} \iiiint_{I_{\eta_0}^M} \left[\frac{m(z)^4 m(z')^4}{1 - m(z)m(z')} + \frac{\sigma_2^3 m(z)^4 m(z')^4}{1 - \sigma_2 m(z)m(z')}\right.$$

$$\left. + (\sigma_4 - 1)m(z)^3 m(z')^3\right] \mathrm{d}\boldsymbol{\eta} \, \mathrm{d}f(\boldsymbol{x}) + \mathcal{O}\left(\iiiint_{I_{\eta_0}^M} \frac{\Psi}{\sqrt{N}\Phi} \, \mathrm{d}\boldsymbol{\eta} \, \mathrm{d}f(\boldsymbol{x})\right),$$

where $\mathrm{d}\boldsymbol{\eta} = \mathrm{d}\eta \, \mathrm{d}\eta'$ and $\mathrm{d}f(\boldsymbol{x}) = \mathrm{d}f(x) \, \mathrm{d}f(x')$. To estimate the error term we have to compute

$$\iint_{-2}^{2} \iint_{\eta_0}^{M} \frac{1}{\eta + \eta' + |x - x'|^2} \frac{1}{\sqrt{\eta\eta'}} \left(\frac{1}{\sqrt{\eta}} + \frac{1}{\sqrt{\eta'}} + \frac{1}{\sqrt{N}\eta\eta'}\right) \mathrm{d}\boldsymbol{\eta} \, \mathrm{d}f(\boldsymbol{x})$$

and readily check that

$$\iiiint_{I_{\eta_0}^M} \frac{\Psi}{\sqrt{N}\Phi} \, \mathrm{d}\boldsymbol{\eta} \, \mathrm{d}f(\boldsymbol{x}) \prec \begin{cases} (|\log M| + |\log \eta_0|)/\sqrt{N} & \text{if } f' \text{ is bounded,} \\ (|\log M| + |\log \eta_0|)/\sqrt{N\eta_0} & \text{else.} \end{cases}$$

By using Lemma 3.2 and organizing the contributions from the boundary terms at η_0 and $-\eta_0$, we find that the leading order of $\mathbf{E}(\widehat{\Delta}_f^{(N)})^2$ becomes

$$\frac{1}{2\pi^2} \Re \iint_{-3}^{3} f(x)f(x') \left(\left[\frac{m(z_0)^4 m(\overline{z_0'})^4}{1 - m(z_0)m(\overline{z_0'})} + \frac{\sigma_2^3 m(z_0)^4 m(\overline{z_0'})^4}{1 - \sigma_2 m(z_0)m(\overline{z_0'})} + (\sigma_4 - 1)m(z_0)^3 m(\overline{z_0'})^3\right]\right.$$

$$(4.11)$$

$$\left. - \left[\frac{m(z_0)^4 m(z_0')^4}{1 - m(z_0)m(z_0')} + \frac{\sigma_2^3 m(z_0)^4 m(z_0')^4}{1 - \sigma_2 m(z_0)m(z_0')} + (\sigma_4 - 1)m(z_0)^3 m(z_0')^3\right]\right) \mathrm{d}\boldsymbol{x} + \mathcal{O}_\prec\left(\frac{\|f\|_1}{M^3}\right),$$

where $z_0 = x + i\eta_0$ and $z_0' = x' + i\eta_0$. Since

$$\frac{a^4}{1 - a} = \frac{a}{1 - a} - a - a^2 - a^3$$

and for any fixed $k \in \mathbb{N}$

$$\frac{1}{2\pi^2}\Re \iint_{-3}^{3} f(x)f(x')\left[m(z_0)^k m(\overline{z_0'})^k - m(z_0)^k m(z_0')^k\right]\mathrm{d}\boldsymbol{x}$$
$$= \left(\frac{1}{\pi}\Im \int_{-2}^{2} f(x)m(x)^k \,\mathrm{d}x\right)^2 + \mathcal{O}_{\prec}(\eta_0)$$

we can conclude that (4.11) becomes

$$\frac{1}{2\pi^2}\Re \iint_{-3}^{3} f(x)f(x')\left(\frac{m(z_0)m(\overline{z_0'})}{1 - m(z_0)m(\overline{z_0'})} - \frac{m(z_0)m(z_0')}{1 - m(z_0)m(z_0')}\right)\mathrm{d}\boldsymbol{x}$$

$$+ \frac{1}{2\pi^2}\Re \iint_{-3}^{3} f(x)f(x')\left(\frac{m(z_0)m(\overline{z_0'})}{1 - \sigma_2 m(z_0)m(\overline{z_0'})} - \frac{m(z_0)m(z_0')}{1 - \sigma_2 m(z_0)m(z_0')}\right)\mathrm{d}\boldsymbol{x}$$

$$- 2\left(\frac{1}{\pi}\Im \int_{\mathbb{R}} f(x)m(x)\,\mathrm{d}x\right)^2 - (1 + \sigma_2)\left(\frac{1}{\pi}\Im \int_{\mathbb{R}} f(x)m(x)^2 \,\mathrm{d}x\right)^2$$

$$(4.12) \qquad + (\sigma_4 - 2 - \sigma_2^2)\left(\frac{1}{\pi}\Im \int_{\mathbb{R}} f(x)m(x)^3 \,\mathrm{d}x\right)^2 + \mathcal{O}\left(\frac{\|f\|_{L^1}}{M^3} + \eta_0\right).$$

The first term of (4.12) was already computed on page 17 of [4]. The computation of the second term is very similar to the first one and the remaining terms are routine calculations. We arrive at

$$\mathbf{E}\left(\widehat{\Delta}_f^{(N)}\right)^2 = V_{f,1} + V_{f,1}^{(\sigma_2)} - 2V_{f,2} - (1 + \sigma_2)V_{f,3} + (\sigma_4 - 2 - \sigma_2^2)V_{f,4}$$
$$+ \mathcal{O}\left(\eta_0 + \frac{\|f\|_1}{M^3} + \frac{|\log M| + |\log \eta_0|}{\sqrt{N \eta_0}}\|\mathrm{d}f\|\right)$$

in the general case and

$$\mathbf{E}\left(\widehat{\Delta}_f^{(N)}\right)^2 = V_{f,1} + V_{f,1}^{(\sigma_2)} - 2V_{f,2} - (1 + \sigma_2)V_{f,3} + (\sigma_4 - 2 - \sigma_2^2)V_{f,4}$$
$$+ \mathcal{O}\left(\eta_0 + \frac{\|f\|_1}{M^3} + \frac{|\log M| + |\log \eta_0|}{\sqrt{N}}\|f'\|_{L^\infty}\right)$$

in the case of f with bounded derivative $f' \in L^\infty([-3,3])$, where

$$V_{f,1} := \int f(x)^2 \,\mu_{sc}(\mathrm{d}x),$$

$$V_{f,1}^{(\sigma_2)} := \iint \frac{f(x)f(y)(1 - \sigma_2^2)}{1 - xy\sigma_2 + (x^2 + y^2 - 2)\sigma_2^2 - xy\sigma_2^3 + \sigma_2^4}\,\mu_{sc}(\mathrm{d}x)\,\mu_{sc}(\mathrm{d}y),$$

$$V_{f,2} := \left(\int f(x)\,\mu_{sc}(\mathrm{d}x)\right)^2, \quad V_{f,3} := \left(\int f(x)x\,\mu_{sc}(\mathrm{d}x)\right)^2,$$

$$(4.13) \qquad V_{f,4} := \left(\int f(x)(x^2 - 1)\,\mu_{sc}(\mathrm{d}x)\right)^2.$$

We note that $V_{f,1}^{(\sigma_2)}$ simplifies to $V_{f,1}^{(1)} = V_{f,1}$ and $V_{f,1}^{(0)} = V_{f,2}$ in the two important cases $\sigma_2 = 0, 1$.

We now choose $M = N$ and η_0 depending on the regularity of f. In the general case of $f \in BV([-3,3])$ it turns out that $\eta_0 = N^{-2/3}$ minimizes the error of $\mathbf{E}\left(\widehat{\Delta}_f^{(N)}\right)^2$, whereas for f with

bounded derivative, a choice of $\eta_0 = N^{-1+\epsilon}$ for any small $\epsilon > 0$ is optimal. Thus

$$(4.14) \qquad \mathbf{E}\left(\widehat{\Delta}_f^{(N)}\right)^2 = \mathbf{E}\left(\widehat{\Delta}_f\right)^2 + \begin{cases} \mathcal{O}_{\prec}\left(N^{-1/2}\right) & \text{if } f' \in L^\infty([-3,3]), \\ \mathcal{O}_{\prec}\left(N^{-1/6}\right) & \text{else.} \end{cases}$$

where $\widehat{\Delta}_f$ is a centered Gaussian of variance

$$\mathbf{E}\left(\widehat{\Delta}_f\right)^2 = V_{f,1} + V_{f,1}^{(\sigma_2)} - 2V_{f,2} - (1+\sigma_2)V_{f,3} + (\sigma_4 - 2 - \sigma_2^2)V_{f,4}.$$

For higher moments we recall the following Wick type factorization Lemma from [4].

Lemma 4.2. *For $k \geq 2$ and $z_1, \ldots, z_k \in \mathbb{C}$ with $z_l = x_l \pm i\eta_l$ and $\eta_l > 0$ we have that*

$$(4.15) \quad \mathbf{E}[X(z_1)\ldots X(z_k)] = \sum_{\pi \in P_2([k])} \prod_{\{a,b\} \in \pi} \mathbf{E}[X(z_a)X(z_b)] + \mathcal{O}_{\prec}\left(\frac{1}{\sqrt{N\eta}}\sum_{a \neq b}\frac{1}{\sqrt{\eta_a}}\Phi_{a,b}\right),$$

where $[k] := \{1, \ldots, k\}$, $\boldsymbol{\eta} := \eta_1 \ldots \eta_k$, $P_2(L)$ are the partitions of a set L into subsets of size 2 and

$$\Phi_{a,b} := \mathbf{1}_{|x_a|,|x_b| \leq 2}\left(|\eta_a| + |\eta_b| + |x_a - x_b|^2\right) + [(|x_a| - 2)_+ + (|x_b| - 2)_+].$$

The error term in (4.15) is slightly stronger than that in [4] since the $\Phi_{a,b}$ includes a $|x_a - x_b|^2$. This strengthening follows along the lines of the original proof by using the more precise analysis of the self consistent equation outlined in Lemma 4.1. We check that integrating the error term from (4.15) over $(I_{\eta_0}^M)^k$, with η_0 being chosen as above according to the regularity of f, again gives asymptotically $N^{-1/2}$ in the case of bounded f' and $N^{-1/6}$ in the general case. By integrating the Wick type product and using (4.14) we therefore arrive at

$$(4.16) \qquad \mathbf{E}\left(\widehat{\Delta}_f^{(N)}\right)^k = \mathbf{E}\left(\widehat{\Delta}_f\right)^k + \begin{cases} \mathcal{O}_{\prec}\left(N^{-1/2}\right) & \text{if } f' \in L^\infty([-3,3]), \\ \mathcal{O}_{\prec}\left(N^{-1/6}\right) & \text{else.} \end{cases}$$

We note that the error terms are implicitly k-dependent. By counting the number of pair partitions we find that, to the leading order in N, the implicit coefficients scale like $C^k(k/2)!$ with a constant depending on f.

Recalling (4.6) and the definition of $T_f^{(N)}$ from (2.6), we conclude that the overall fluctuations have moments

$$(4.17) \qquad \mathbf{E}\left(T_f^{(N)}\right)^k = \mathbf{E}\left(\widehat{\Delta}_f\right)^k + \begin{cases} \mathcal{O}\left(C^k(k/2)!N^{-1/2}\right) & \text{if } f' \in L^\infty([-3,3]), \\ \mathcal{O}\left(C^k(k/2)!N^{-1/6}\right) & \text{else.} \end{cases}$$

Let $\phi_N(t)$ denote the characteristic function of $T_f^{(N)}$ and $\phi(t)$ the characteristic function of the Gaussian variable $\widehat{\Delta}_f$. Then the moment bound (4.17) implies that

$$|\phi_N(t) - \phi(t)| \leq CN^{-1/6}te^{Ct^2}$$

with some constant C depending on f. Using the well-known bound (see, e.g., [18, Theorem 1.4.13.] and the references therein)

$$d_L(F, G) \leq \frac{1}{\pi}\int_0^T |\phi_F(t) - \phi_G(t)|\frac{\mathrm{d}t}{t} + \frac{2e\log T}{T}$$

for any two distributions F and G with characteristic functions ϕ_F and ϕ_G, we immediately obtain (2.7) by choosing $T = c\sqrt{\log N}$. This completes the proof of Theorem 2.2.

5. Off-diagonal entries

For the decomposition

$$H = \begin{pmatrix} h_{11} & h_{12} & h_1^* \\ h_{21} & h_{22} & h_2^* \\ h_1 & h_2 & \widehat{H} \end{pmatrix}$$

we find from the Schur complement formula that

$$G(z)_{12} = -\frac{g_{12}}{g_{11}g_{22} - g_{12}g_{21}} = -m(z)^2 g_{12} + \mathcal{O}_\prec\left(\frac{1}{N\,|\eta|}\right),$$

where $g_{ij} := h_{ij} - \delta_{ij}z - \langle h_i, G(z)h_j\rangle$.

We now set $Y(z) = Y^{(N)}(z) := \sqrt{N}\left\langle h_1, \widehat{G}(z)h_2\right\rangle$ and begin to compute (all summation indices run from 3 to N)

(5.1)

$$\mathbf{E}\left[Y(z)Y(z')|\widehat{H}\right] = N\sum_{a,b,c,d}\mathbf{E}\left[h_{1a}\widehat{G}_{ab}h_{b2}h_{1c}\widehat{G}'_{cd}h_{d2}|\widehat{H}\right]$$

$$= \frac{\sigma_2^2}{N}\sum_{a,b}\widehat{G}_{ab}\widehat{G}'_{ab} + \mathcal{O}_\prec\left(\frac{\Psi}{N}\right) = \frac{\sigma_2^2 m(z)m(z')}{1 - \sigma_2 m(z)m(z')} + \mathcal{O}_\prec\left(\frac{\Psi}{\sqrt{N}\Phi}\right)$$

and

(5.2) $$\mathbf{E}\left[Y(z)\overline{Y(z')}|\widehat{H}\right] = N\sum_{a,b,c,d}\mathbf{E}\left[h_{1a}\widehat{G}_{ab}h_{b2}h_{2c}\widehat{G}'_{cd}h_{d1}|\widehat{H}\right]$$

$$= \frac{1}{N}\sum_{a,b}\widehat{G}_{ab}\widehat{G}'_{ba} + \mathcal{O}_\prec\left(\frac{\Psi}{N}\right) = \frac{m(z)m(z')}{1 - m(z)m(z')} + \mathcal{O}_\prec\left(\frac{\Psi}{\sqrt{N}\Phi}\right).$$

For both estimates we made use of the fact the h_{ab} are centered and therefore have to appear at least twice to have non-zero expectation. The main contribution comes from the pairing $a = d$, $b = c$. Some exceptional pairings, such as the four-pairing $a = b = c = d$, were incorporated in the error term by their reduced combinatorics. From Proposition 3.1 we then find that

$$f(H)_{12} = \frac{1}{\pi}\iint_{I_{\eta_0}^M} m(z)^2\left[\left\langle h_1, \widehat{G}(z)h_2\right\rangle - h_{12}\right]\mathrm{d}\eta\,\mathrm{d}f(x) + \mathcal{O}_\prec\left(\frac{\|\mathrm{d}f\|}{N}\right).$$

For the second term it follows, just as before, that

$$\frac{1}{\pi}\iint_{I_{\eta_0}^M} m(z)^2 h_{12}\,\mathrm{d}\eta\,\mathrm{d}f(x) = h_{12}\int f(x)x\,\mu_{sc}(\mathrm{d}x) + \mathcal{O}_\prec(\eta_0).$$

For the first term we set

$$\widetilde{\Delta}_f^{(N)} := \iint_{I_{\eta_0}^M} m(z)^2 Y(z)\,\mathrm{d}\eta\,\mathrm{d}f(x)$$

and by a computation analogous to (4.11) using (5.1) and an expansion of the form

$$\frac{a^3}{1-a} = \frac{a}{1-a} - a - a^2$$

we arrive at

$$\mathbf{E}\left(\widetilde{\Delta}_f^{(N)}\right)^2 = V_{f,1}^{(\sigma_2)} - V_{f,2} - \sigma_2 V_{f,3} + \begin{cases} \mathcal{O}_\prec\left(N^{-1/2}\right) & \text{if } f' \in L^\infty([-3,3]), \\ \mathcal{O}_\prec\left(N^{-1/6}\right) & \text{else.} \end{cases}$$

Similarly, from (5.2) we find that

$$\mathbf{E}\left|\widetilde{\Delta}_f^{(N)}\right|^2 = V_{f,1} - V_{f,2} - V_{f,3} + \begin{cases} \mathcal{O}_\prec\left(N^{-1/2}\right) & \text{if } f' \in L^\infty([-3,3]), \\ \mathcal{O}_\prec\left(N^{-1/6}\right) & \text{else.} \end{cases}$$

Finally, due to a Wick type theorem for $Y(z)$ which is proved along the lines of Lemma 4.2 we arrive at

$$(5.3) \qquad \mathbf{E} \left(S_f^{(N)} \right)^k \left(\overline{S_f^{(N)}} \right)^l = \mathbf{E} \left(\widetilde{\Delta}_f \right)^k \left(\overline{\widetilde{\Delta}_f} \right)^l + \begin{cases} \mathcal{O}_\prec \left(N^{-1/2} \right) & \text{if } f' \in L^\infty([-3,3]), \\ \mathcal{O}_\prec \left(N^{-1/6} \right) & \text{else,} \end{cases}$$

where $\widetilde{\Delta}_f$ is a centered complex Gaussian such that

$$\mathbf{E} \, \widetilde{\Delta}_f^2 = V_{f,1}^{(\sigma_2)} - V_{f,2} - \sigma_2 V_{f,3}, \quad \mathbf{E} \left| \widetilde{\Delta}_f \right|^2 = V_{f,1} - V_{f,2} - V_{f,3}.$$

We have proven Theorem 2.3.

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